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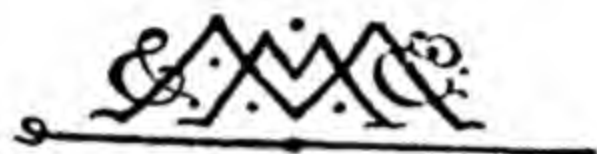
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# SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S  
ELEMENTARY TRIGONOMETRY.

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# SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S

ELEMENTARY TRIGONOMETRY

BY

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## PREFACE.

IN preparing this Key two objects have been kept in view. It is intended first to save the time and lighten the work of teachers, and secondly to afford help to those who study Mathematics without the guidance of a teacher. Accordingly the solutions have generally been given in the most simple and natural manner, with frequent reference to the text and examples in the *Elementary Trigonometry*. In particular, the solutions which involve logarithmic work have been presented in the fullest detail, so that with the help of the Key, a teacher will be able very readily to discover and correct mistakes in the work of his pupils.

For very many of the solutions I am indebted to Mr H. C. Playne of Clifton College, and my thanks are due to him for valuable help all through the book.

H. S. HALL.

January, 1895.

THE present Edition contains solutions of all the examples introduced into the Fourth Edition of the *Elementary Trigonometry*. For many of these I am indebted to Mr H. C. Beaven of Clifton College, whose valuable help I gratefully acknowledge.

H. S. HALL.

October, 1905.



# ELEMENTARY TRIGONOMETRY.

## EXAMPLES. I. PAGE 4.

7.  $69^{\circ} 13' 30'' = \cdot 76916$  of a right angle  $= 76^{\circ} 91' 66\cdot 7''$ .
8.  $19^{\circ} 0' 45'' = \cdot 21125$  of a right angle  $= 21^{\circ} 12' 50''$ .
9.  $50^{\circ} 37' 5\cdot 7'' = \cdot 562425$  of a right angle  $= 56^{\circ} 24' 25''$ .
10.  $43^{\circ} 52' 38\cdot 1'' = \cdot 487525$  of a right angle  $= 48^{\circ} 75' 25''$ .
11.  $11^{\circ} 0' 38\cdot 4'' = \cdot 1223407$  of a right angle  $= 12^{\circ} 23' 40\cdot 7''$ .
12.  $142^{\circ} 15' 45'' = 1\cdot 5806944$  of a right angle  $= 158^{\circ} 6' 94\cdot 4''$ .
13.  $12' 9'' = \cdot 00225$  of a right angle  $= 22' 50''$ .
14.  $3' 26\cdot 3'' = \cdot 000636$  of a right angle  $= 6' 36\cdot 7''$ .
15.  $56^{\circ} 87' 50'' = \cdot 56875$  of a right angle  $= 51^{\circ} 11' 15''$ .
16.  $39^{\circ} 6' 25'' = \cdot 390625$  of a right angle  $= 35^{\circ} 9' 22\cdot 5''$ .
17.  $40^{\circ} 1' 25\cdot 4'' = \cdot 4001254$  of a right angle  $= 36^{\circ} 0' 40\cdot 6''$ .
18.  $1^{\circ} 2' 3'' = \cdot 010203$  of a right angle  $= 55' 5\cdot 8''$ .
19.  $3^{\circ} 2' 55'' = \cdot 030205$  of a right angle  $= 2^{\circ} 43' 6\cdot 4''$ .
20.  $8^{\circ} 10' 6\cdot 5'' = \cdot 0810065$  of a right angle  $= 7^{\circ} 17' 26\cdot 1''$ .
21.  $6' 25'' = \cdot 000625$  of a right angle  $= 3' 22\cdot 5''$ .
22.  $37' 5'' = \cdot 003705$  of a right angle  $= 20' 0\cdot 4''$ .
23. Let the angles expressed in degrees be  $A$  and  $B$ ;

then  $A + B = \frac{9}{10} \times 80^{\circ} = 72^{\circ}$ , and  $A - B = 18^{\circ}$ .

Hence  $A = 45^{\circ}$ ,  $B = 27^{\circ}$

24. If  $n$  is the number of degrees in the angle,  $n + \frac{10}{9}n = 152$ ; whence  $n = 72$ .

25. Here  $\frac{x}{60}$  = number of degrees, and  $\frac{y}{100}$  = number of grades in the angle. Therefore  $\frac{x}{60} = \frac{9}{10} \cdot \frac{y}{100}$ ; whence we obtain  $50x = 27y$ .

26. Here  $\frac{s}{60 \times 60}$  = number of degrees, and  $\frac{t}{100 \times 100}$  = number of grades in the angle. Therefore  $\frac{s}{36} = \frac{9}{10} \times \frac{t}{100}$ ; that is,  $250s = 81t$ .

### EXAMPLES. II. PAGE 11.

[The following five solutions will sufficiently illustrate this exercise.]

3. From fig. of Art. 17 we have  $a^2 = b^2 + c^2 = 400 + 225 = 625$ ; whence  $a = 25$ , and  $\sin C = \frac{4}{5}$ ,  $\cos B = \frac{4}{5}$ ,  $\cot C = \frac{3}{4}$ ,  $\sec C = \frac{5}{3}$ .

6. Let  $a = 15$ ,  $b = 9$ ; then  $c^2 = a^2 - b^2 = (a + b)(a - b) = 24 \times 6$ ; whence  $c = 12$ , and  $\sin C = \frac{4}{5}$ ,  $\cos C = \frac{3}{5}$ ,  $\tan C = \frac{4}{3}$ .

8. In the third fig. of Art. 17, let  $AC = 41$ ,  $AB = 9$ ; then  $BC^2 = 41^2 - 9^2 = (41 + 9)(41 - 9)$ ; whence  $BC = 40$ , and  $\sin A = \frac{40}{41}$ ,  $\cot A = \frac{9}{40}$ .

10. Here  $CD = 2DE$ . Hence if  $ED = a$ ,  $DC = 2a$ ,  $EC = a\sqrt{5}$ . The required ratios may now be written down.

11. From the right-angled  $\triangle ABC$  we have  $BC = 39$ ; also from the right-angled  $\triangle ACD$  we have  $DC = 77$ . The required ratios may now be written down.

### EXAMPLES. III a. PAGE 17.

Examples 1—25 are too easy to require full solution; the following eight solutions will suffice.

$$10. (1 - \cos^2 \theta) \sec^2 \theta = \sin^2 \theta \times \frac{1}{\cos^2 \theta} = \tan^2 \theta.$$

$$12. \operatorname{cosec} a \sqrt{1 - \sin^2 a} = \frac{1}{\sin a} \times \cos a = \cot a.$$

$$15. (1 - \cos^2 \theta) (1 + \tan^2 \theta) = \sin^2 \theta \sec^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta.$$

$$19. (1 - \cos^2 A) (1 + \cot^2 A) = \sin^2 A \operatorname{cosec}^2 A = 1.$$

$$20. \sin a \sec a \sqrt{\operatorname{cosec}^2 a - 1} = \frac{\sin a}{\cos a} \times \cot a = 1.$$

$$22. \sin^2 \theta \cot^2 \theta + \sin^2 \theta = \sin^2 \theta (1 + \cot^2 \theta) = \sin^2 \theta \operatorname{cosec}^2 \theta = 1.$$

$$23. (1 + \tan^2 \theta) (1 - \sin^2 \theta) = \sec^2 \theta \cos^2 \theta = 1.$$

$$25. \operatorname{cosec}^2 \theta \tan^2 \theta - 1 = \frac{1}{\sin^2 \theta} \cdot \frac{\sin^2 \theta}{\cos^2 \theta} - 1 = \sec^2 \theta - 1 = \tan^2 \theta.$$

$$26. \text{First side} = \cos^2 A + \sin^2 A = 1.$$

$$27. \text{First side} = \sec^2 A - \tan^2 A = 1.$$

$$28. \text{First side} = \sin A \cdot \sin A + \cos A \cdot \cos A = \sin^2 A + \cos^2 A = 1.$$

$$29. \text{First side} = \sec A \cdot \sec A - \tan A \cdot \tan A = \sec^2 A - \tan^2 A = 1.$$

$$\begin{aligned} 30. \sin^4 a - \cos^4 a &= (\sin^2 a + \cos^2 a) (\sin^2 a - \cos^2 a) \\ &= \sin^2 a - \cos^2 a = \sin^2 a - (1 - \sin^2 a) \\ &= 2 \sin^2 a - 1. \end{aligned}$$

$$\text{Also } \sin^2 a - \cos^2 a = 1 - \cos^2 a - \cos^2 a = 1 - 2 \cos^2 a.$$

$$31. \text{First side} = (\sec^2 a - 1) (\sec^2 a + 1) = \tan^2 a (2 + \tan^2 a).$$

$$32. \text{First side} = (\operatorname{cosec}^2 a - 1) (\operatorname{cosec}^2 a + 1) = \cot^2 a (\cot^2 a + 2).$$

$$33. \text{First side} = \left( \frac{\sin a}{\cos a} \cdot \frac{1}{\sin a} \right)^2 - \left( \frac{\sin a}{\cos a} \right)^2 = \sec^2 a - \tan^2 a.$$

$$34. \text{First side} = \left( \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \right)^2 - \left( \frac{\cos \theta}{\sin \theta} \right)^2 = \operatorname{cosec}^2 \theta - \cot^2 \theta.$$

$$35. \operatorname{cosec}^2 \theta - \cot^2 \theta = 1 = \sec^2 \theta - \tan^2 \theta. \quad \text{Transpose.}$$

### EXAMPLES. III b. PAGE 19.

$$4. \operatorname{vers} \theta \sec \theta = (1 - \cos \theta) \sec \theta = \sec \theta - 1.$$

$$5. \text{First side} = \frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \sin \theta = \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} = \cos \theta.$$

$$\begin{aligned} 6. \text{First side} &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\sin \theta \cos \theta} = \operatorname{cosec} \theta \sec \theta. \end{aligned}$$

$$7. \text{First side} = \operatorname{cosec} A \tan A \cos A = \frac{\cos A}{\sin A} \tan A = 1.$$

$$\begin{aligned} 8. \text{First side} &= \cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta \\ &= 1 + 1 = 2. \end{aligned}$$

$$\begin{aligned} 9. \text{First side} &= 1 + 2 \tan \theta + \tan^2 \theta + 1 - 2 \tan \theta + \tan^2 \theta \\ &= 2 + 2 \tan^2 \theta = 2 (1 + \tan^2 \theta) = 2 \sec^2 \theta. \end{aligned}$$

$$10. \text{ First side} = \cot^2 \theta - 2 \cot \theta + 1 + \cot^2 \theta + 2 \cot \theta + 1 \\ = 2 \cot^2 \theta + 2 = 2 (\cot^2 \theta + 1) = 2 \operatorname{cosec}^2 \theta.$$

$$11. \text{ First side} = \sin^2 A \operatorname{cosec}^2 A + \cos^2 A \sec^2 A = 1 + 1 = 2.$$

$$12. \text{ First side} = \cos^2 A \times 1 + \sin^2 A \times 1 = 1.$$

$$13. \text{ First side} = \cot^2 a (1 + \cot^2 a) = (\operatorname{cosec}^2 a - 1) \operatorname{cosec}^2 a \\ = \operatorname{cosec}^4 a - \operatorname{cosec}^2 a.$$

$$14. \text{ First side} = \frac{\tan^2 a}{\sec^2 a} \cdot \frac{\operatorname{cosec}^2 a}{\cot^2 a} = \frac{\sin^2 a}{\cos^2 a \sec^2 a} \times \frac{\operatorname{cosec}^2 a \sin^2 a}{\cos^2 a} \\ = \frac{\sin^2 a}{\cos^2 a} = \sin^2 a \sec^2 a.$$

$$15. \text{ First side} = \frac{1 + \sin a + 1 - \sin a}{1 - \sin^2 a} = \frac{2}{\cos^2 a} = 2 \sec^2 a.$$

$$16. \text{ First side} = \frac{\tan a (\sec a + 1) + \tan a (\sec a - 1)}{\sec^2 a - 1} = \frac{2 \tan a \sec a}{\tan^2 a} \\ = \frac{2 \sec a}{\tan a} = \frac{2 \sec a \cos a}{\sin a} = \frac{2}{\sin a} = 2 \operatorname{cosec} a.$$

$$17. \text{ First side} = \frac{1}{1 + \sin^2 a} + \frac{1}{1 + \frac{1}{\sin^2 a}} = \frac{1}{1 + \sin^2 a} + \frac{\sin^2 a}{1 + \sin^2 a} \\ = \frac{1 + \sin^2 a}{1 + \sin^2 a} = 1.$$

$$18. \text{ First side} = \left( \frac{1}{\cos \theta} + \frac{1}{\sin \theta} \right) (\sin \theta + \cos \theta) \\ = \frac{(\sin \theta + \cos \theta) (\sin \theta + \cos \theta)}{\cos \theta \sin \theta} \\ = \frac{\sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta}{\cos \theta \sin \theta} = \frac{1 + 2 \cos \theta \sin \theta}{\cos \theta \sin \theta} \\ = \frac{1}{\cos \theta \sin \theta} + 2 = \sec \theta \csc \theta + 2.$$

$$20. \text{ First side} = (1 + \cot \theta)^2 - \operatorname{cosec}^2 \theta = 1 + 2 \cot \theta + \cot^2 \theta - \operatorname{cosec}^2 \theta \\ = 1 + 2 \cot \theta - 1 = 2 \cot \theta.$$

$$22. \text{ First side} \\ = \sin^2 A + 2 \sin A \operatorname{cosec} A + \operatorname{cosec}^2 A + \cos^2 A + 2 \cos A \sec A + \sec^2 A \\ = (\sin^2 A + \cos^2 A) + 2 + (\cot^2 A + 1) + 2 + (\tan^2 A + 1) \\ = \tan^2 A + \cot^2 A + 7.$$

$$23. \text{ First side} = (2 \sec^2 A - 1) (2 \operatorname{cosec}^2 A - 1) \\ = 4 \sec^2 A \operatorname{cosec}^2 A - 2 \sec^2 A - 2 \operatorname{cosec}^2 A + 1 \\ = 1 + 4 \sec^2 A \operatorname{cosec}^2 A - 2 (\sec^2 A \operatorname{cosec}^2 A) \quad [\text{Art. 31, Ex. 1.}] \\ = 1 + 2 \sec^2 A \operatorname{cosec}^2 A.$$

$$\begin{aligned}
 24. \quad \text{First side} &= 1 + (\sin^2 A + \cos^2 A) - 2 \sin A + 2 \cos A - 2 \sin A \cos A \\
 &= 2 [1 - \sin A + \cos A - \sin A \cos A] \\
 &= 2 (1 - \sin A) (1 + \cos A).
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \text{First side} &= \sin A \left(1 + \frac{\sin A}{\cos A}\right) + \cos A \left(1 + \frac{\cos A}{\sin A}\right) \\
 &= \sin A \frac{(\cos A + \sin A)}{\cos A} + \cos A \frac{(\sin A + \cos A)}{\sin A} \\
 &= \tan A (\sin A + \cos A) + \cot A (\sin A + \cos A) \\
 &= (\sin A + \cos A) \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right) \\
 &= \frac{(\sin A + \cos A) (\sin^2 A + \cos^2 A)}{\sin A \cos A} \\
 &= \frac{\sin A + \cos A}{\sin A \cos A} = \frac{1}{\cos A} + \frac{1}{\sin A} = \sec A + \operatorname{cosec} A.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \text{First side} &= \cos \theta (2 \tan^2 \theta + 5 \tan \theta + 2) = \frac{2 \sin^2 \theta}{\cos^2 \theta} \cos \theta + 5 \sin \theta + 2 \cos \theta \\
 &= 2 \left(\frac{\sin^2 \theta}{\cos \theta} + \cos \theta\right) + 5 \sin \theta = \frac{2 (\sin^2 \theta + \cos^2 \theta)}{\cos \theta} + 5 \sin \theta \\
 &= 2 \sec \theta + 5 \sin \theta.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \text{First side} &= \left(\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta}\right)^2 = \left(\frac{1 + \sin \theta}{\cos \theta}\right)^2 \\
 &= \frac{(1 + \sin \theta) (1 + \sin \theta)}{1 - \sin^2 \theta} = \frac{(1 + \sin \theta) (1 + \sin \theta)}{(1 + \sin \theta) (1 - \sin \theta)} = \frac{1 + \sin \theta}{1 - \sin \theta}.
 \end{aligned}$$

$$28. \quad \text{First side} = \frac{\cos \theta (2 \sin \theta - 1)}{2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (2 \sin \theta - 1)}{\sin \theta (2 \sin \theta - 1)} = \cot \theta.$$

$$\begin{aligned}
 29. \quad \text{First side} &= \frac{\cot^2 \theta (\sec \theta - 1)}{1 + \sin \theta} - \frac{\sec^2 \theta (1 - \sin \theta)}{1 + \sec \theta} \\
 &= \frac{\cot^2 \theta (\sec^2 \theta - 1) - \sec^2 \theta (1 - \sin^2 \theta)}{(1 + \sin \theta) (1 + \sec \theta)} \\
 &= \frac{\cot^2 \theta \tan^2 \theta - \sec^2 \theta \cos^2 \theta}{(1 + \sin \theta) (1 + \sec \theta)} = 0.
 \end{aligned}$$

$$30. \quad \tan^2 \alpha + \sec^2 \beta = (\sec^2 \alpha - 1) + (\tan^2 \beta + 1) = \sec^2 \alpha + \tan^2 \beta.$$

$$31. \quad \frac{\tan \alpha + \cot \beta}{\cot \alpha + \tan \beta} = \frac{\tan \alpha + \frac{1}{\tan \beta}}{\frac{1}{\tan \alpha} + \tan \beta} = \frac{\tan \alpha \tan \beta + 1}{\tan \beta} \cdot \frac{\tan \alpha}{\tan \alpha \tan \beta + 1} = \frac{\tan \alpha}{\tan \beta}.$$

$$\begin{aligned}
 33. \quad \text{First side} &= \cot \alpha \tan \alpha \tan \beta + \tan \beta \cot \beta \cot \alpha \\
 &= \tan \beta + \cot \alpha.
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \text{First side} &= \sin^2 \alpha (1 - \sin^2 \beta) - (1 - \sin^2 \alpha) \sin^2 \beta \\
 &= \sin^2 \alpha - \sin^2 \alpha \sin^2 \beta - \sin^2 \beta + \sin^2 \alpha \sin^2 \beta \\
 &= \sin^2 \alpha - \sin^2 \beta.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \text{First side} &= (1 + \tan^2 \alpha) \tan^2 \beta - \tan^2 \alpha (1 + \tan^2 \beta) \\
 &= \tan^2 \beta + \tan^2 \alpha \tan^2 \beta - \tan^2 \alpha - \tan^2 \alpha \tan^2 \beta \\
 &= \tan^2 \beta - \tan^2 \alpha.
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \text{First side} &= \sin^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta + \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \sin^2 \beta, \\
 &\text{the other terms cancelling;}
 \end{aligned}$$

$$\begin{aligned}
 \text{this expression} &= (\sin^2 \alpha + \cos^2 \alpha) \cos^2 \beta + (\cos^2 \alpha + \sin^2 \alpha) \sin^2 \beta \\
 &= \cos^2 \beta + \sin^2 \beta = 1.
 \end{aligned}$$

### EXAMPLES. III. c. PAGE 23.

$$1. \quad \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}} = \frac{1}{\sqrt{1 - \frac{1}{4}}} = \frac{2}{\sqrt{3}}.$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}.$$

$$2. \quad \sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} \quad [\text{Art. 32, Ex. 1}]$$

$$= \frac{4}{3} \div \sqrt{1 + \frac{16}{9}} = \frac{4}{3} \times \frac{3}{5} = \frac{4}{5}.$$

$$\cos A = \cot A \cdot \sin A = \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}.$$

$$5. \quad \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{49}} = \sqrt{\frac{48}{49}} = \frac{\sqrt{48}}{7}.$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{7} \div \frac{\sqrt{48}}{7} = \frac{1}{\sqrt{48}}.$$

$$6. \quad \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{7^2}{25^2}} = \frac{24}{25}. \quad \text{Therefore } \sec A = \frac{25}{24}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{7}{25} \times \frac{25}{24} = \frac{7}{24}.$$

$$8. \quad \operatorname{cosec} \alpha = \sqrt{1 + \cot^2 \alpha}. \quad [\text{Art. 27.}]$$

$$\cos \alpha = \cot \alpha \sin \alpha = \frac{\cot \alpha}{\operatorname{cosec} \alpha} = \frac{\cot \alpha}{\sqrt{1 + \cot^2 \alpha}}.$$

### III.] RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS. 7

$$10. \operatorname{cosec} A = \frac{1}{\sin A}; \quad \cos A = \sqrt{1 - \sin^2 A}; \quad \sec A = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}; \quad \cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

$$11. \text{ Here } \sin A = \cos A, \text{ so that } \tan A = 1.$$

$$\therefore \operatorname{cosec} A = \sqrt{1 + \cot^2 A} = \sqrt{1 + 1} = \sqrt{2}.$$

$$12. \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{m}{n} \div \sqrt{1 - \frac{m^2}{n^2}}$$

$$= \frac{m}{n} \times \frac{n}{\sqrt{n^2 - m^2}} = \frac{m}{\sqrt{n^2 - m^2}}.$$

$$13. p^2 \cot^2 \theta = q^2 - p^2; \quad \therefore p^2 (\cot^2 \theta + 1) = q^2.$$

$$\therefore p^2 \operatorname{cosec}^2 \theta = q^2, \text{ so that } \sin \theta = \frac{p}{q}.$$

$$14. \text{ In the diagram of Ex. 2, Art. 33, let } PQ = 2m, PR = m^2 + 1; \text{ then}$$

$$RQ^2 = (m^2 + 1)^2 - (2m)^2 = (m^2 - 1)^2.$$

$$\therefore RQ = m^2 - 1.$$

$$\therefore \tan A = \frac{m^2 - 1}{2m}, \quad \sin A = \frac{m^2 - 1}{m^2 + 1}.$$

$$16. \text{ The expression} = \frac{2 \tan a - 3}{4 \tan a - 9}; \text{ but } \tan a = \sqrt{\frac{169}{25}} - 1 = \frac{12}{5}.$$

$$\therefore \text{ the expression} = \frac{\frac{2 \cdot 12}{5} - 3}{\frac{4 \cdot 12}{5} - 9} = \frac{9}{3} = 3.$$

$$17. \text{ The expression} = \frac{p \cot \theta - q}{p \cot \theta + q} = \frac{\frac{p^2}{q} - q}{\frac{p^2}{q} + q} = \frac{p^2 - q^2}{p^2 + q^2}.$$

### EXAMPLES. IV. a. PAGE 26.

Let  $E$  stand for the expression to be evaluated in each case; then

$$6. E = (1)^2 \times \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} (\sqrt{3})^2 = \frac{3}{2}.$$

$$7. E = (\sqrt{3})^2 + 4 \left( \frac{1}{\sqrt{2}} \right)^2 + 3 \left( \frac{2}{\sqrt{3}} \right)^2 = 3 + \frac{4}{2} + 4 = 9.$$

8.  $E = \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right)^2 + (\sqrt{2})^2 - 2 \left( \frac{1}{\sqrt{3}} \right)^2 = \frac{1}{2} \cdot \frac{4}{3} + 2 - \frac{2}{3} = 2.$
9.  $E = \left( \frac{1}{\sqrt{3}} \right)^2 + 2 \left( \frac{\sqrt{3}}{2} \right) + 1 - \sqrt{3} + \left( \frac{\sqrt{3}}{2} \right)^2 = \frac{1}{3} + \sqrt{3} + 1 - \sqrt{3} + \frac{3}{4} = \frac{25}{12}.$
10.  $E = (1)^2 + \frac{1}{2} - \left( \frac{\sqrt{3}}{2} \right)^2 - \frac{3}{4} \left( \frac{1}{\sqrt{3}} \right)^2 = 1 + \frac{1}{2} - \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$
11.  $E = 3 \left( \frac{1}{\sqrt{3}} \right)^2 + \frac{4}{3} \left( \frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} (\sqrt{2})^2 - \frac{1}{3} \left( \frac{\sqrt{3}}{2} \right)^2 = 1 + 1 - 1 - \frac{1}{4} = \frac{3}{4}.$
12.  $E = \frac{1}{2} - 1^2 + \frac{3}{4} \left( \frac{1}{\sqrt{3}} \right)^2 + \left( \frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} = \frac{1}{2} - 1 + \frac{1}{4} + \frac{3}{4} - \frac{1}{2} = 0.$
13.  $E = \frac{1}{3} \left( \frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} \cdot 2 \left( \frac{1}{\sqrt{3}} \right)^2 + \frac{4}{3} \left( \frac{1}{\sqrt{2}} \right)^2 (\sqrt{3})^2 = \frac{1}{4} - \frac{1}{3} + 2 = 1\frac{1}{4}.$
14. We have
- $$(1)^2 - \left( \frac{1}{2} \right)^2 = x \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{3};$$
- $$1 - \frac{1}{4} = x \cdot \frac{\sqrt{3}}{2};$$
- $$\therefore x = \frac{\sqrt{3}}{2}.$$
15. We have
- $$x \cdot \frac{1}{2} \cdot \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{(\sqrt{3})^2 \cdot 2 \cdot 1}{(\sqrt{2})^2 \cdot 2};$$
- $$\frac{x}{4} = \frac{3}{2};$$
- $$\therefore x = 6.$$

**EXAMPLES. IV. b. PAGE 28.**

For Examples 9--14, see Example 1, page 28.

20. Second side  $= 1 - \sin^2 A \sec^2 A = 1 + \tan^2 A = \sec^2 A = \operatorname{cosec}^2 (90^\circ - A).$
21. First side  $= \sin A \cot A \tan A \operatorname{cosec} A = 1.$
22. First side  $= \operatorname{cosec} A - \cot A \sin A \cot A = \operatorname{cosec} A (1 - \cos^2 A) = \sin A.$
23. First side  $= \tan^2 A \operatorname{cosec}^2 A - \sin^2 A \sec^2 A$   
 $= \sec^2 A - \tan^2 A = 1.$
24. First side
- $$= \cot A + \tan A = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \operatorname{cosec} A \sec A = \operatorname{cosec} A \operatorname{cosec} (90^\circ - A).$$
25. First side  $= \frac{\cos A}{\operatorname{cosec} A} \cdot \frac{\cot A}{\cos A} = \cos A.$
26. First side  $= \frac{\operatorname{cosec}^2 A \tan^2 A}{\tan A} \cdot \frac{\cot A}{\sec^2 A} = \cot^2 A.$   
 $= \operatorname{cosec}^2 A - 1 = \sec^2 (90^\circ - A) - 1.$

$$27. \text{ First side} = \frac{\tan A}{\operatorname{cosec}^2 A} \cdot \frac{\sec A \cot^3 A}{\cos^2 A} = \sec A = \sqrt{\tan^2 A + 1}.$$

$$28. \text{ First side} = \frac{\sin^2 A}{1 - \cos A} = \frac{1 - \cos^2 A}{1 - \cos A} = 1 + \cos A = 1 + \sin (90^\circ - A).$$

$$29. \text{ First side} = \frac{\cot^2 A \cos^2 A}{\cot A (1 + \sin A)} = \cot A (1 - \sin A) = \tan (90^\circ - A) - \cos A.$$

$$30. \quad x \cos A \tan A = \sin A ; \\ \therefore x = 1.$$

$$31. \quad \sec^2 A - x \tan A = 1 ; \\ \therefore \tan^2 A = x \tan A ; \\ \therefore x = \tan A.$$

### EXAMPLES. IV. d. PAGE 31.

$$9. \quad 1 + \tan^2 \theta = 2 \tan^2 \theta ; \\ \tan^2 \theta = 1 ; \\ \therefore \tan \theta = \pm 1 ; \\ \therefore \theta = 45^\circ.$$

$$10. \quad 1 + \cot^2 \theta = 4 \cot^2 \theta ; \\ 3 \cot^2 \theta = 1 ; \\ \therefore \cot \theta = \pm \frac{1}{\sqrt{3}} ; \\ \therefore \theta = 60^\circ.$$

$$11. \quad 1 + \tan^2 \theta = 3 \tan^2 \theta - 1 ; \\ 2 \tan^2 \theta = 2 ; \\ \therefore \tan \theta = \pm 1 ; \\ \therefore \theta = 45^\circ.$$

$$12. \quad 1 + \tan^2 \theta + \tan^2 \theta = 7 ; \\ 2 \tan^2 \theta = 6 ; \\ \therefore \tan \theta = \pm \sqrt{3} ; \\ \therefore \theta = 60^\circ.$$

$$13. \quad \cot^2 \theta + 1 + \cot^2 \theta = 3 ; \\ \cot^2 \theta = 1 ; \\ \therefore \cot \theta = \pm 1 ; \\ \therefore \theta = 45^\circ.$$

$$14. \quad 2 (\cos^2 \theta - 1 + \cos^2 \theta) = 1 ; \\ 4 \cos^2 \theta = 3 ; \\ \therefore \cos \theta = \pm \frac{\sqrt{3}}{2} ; \\ \therefore \theta = 30^\circ.$$

$$15. \quad 2 \cos^2 \theta + 4 - 4 \cos^2 \theta = 3 ; \\ 2 \cos^2 \theta = 1 ; \\ \therefore \cos \theta = \pm \frac{1}{\sqrt{2}} ; \\ \therefore \theta = 45^\circ.$$

$$16. \quad 6 \cos^2 \theta - \cos \theta - 1 = 0 ; \\ (3 \cos \theta + 1) (2 \cos \theta - 1) = 0 ; \\ \therefore \cos \theta = \frac{1}{2} \text{ or } -\frac{1}{3} ; \\ \therefore \theta = 60^\circ.$$

$$17. \quad 12 \sin^2 \theta - 4 \sin \theta - 1 = 0 ; \\ (6 \sin \theta + 1) (2 \sin \theta - 1) = 0 ; \\ \therefore \sin \theta = \frac{1}{2} \text{ or } -\frac{1}{6} ; \\ \therefore \theta = 30^\circ.$$

$$18. \quad 2 - 2 \cos^2 \theta = 3 \cos \theta ; \\ 2 \cos^2 \theta + 3 \cos \theta - 2 = 0 ; \\ \therefore (2 \cos \theta - 1) (\cos \theta + 2) = 0 ; \\ \therefore \cos \theta = \frac{1}{2}, \text{ so that } \theta = 60^\circ.$$

$$19. \quad \tan \theta = 4 - \frac{3}{\tan \theta};$$

$$\tan^2 \theta - 4 \tan \theta + 3 = 0;$$

$$\therefore (\tan \theta - 1)(\tan \theta - 3) = 0;$$

$$\therefore \tan \theta = 1 \text{ or } 3;$$

$$\therefore \theta = 45^\circ, \text{ or } 71^\circ 34'.$$

$$20. \quad \cos^2 \theta - 1 + \cos^2 \theta = 2 - 5 \cos \theta;$$

$$2 \cos^2 \theta + 5 \cos \theta - 3 = 0;$$

$$\therefore (2 \cos \theta - 1)(\cos \theta + 3) = 0;$$

$$\therefore \cos \theta = \frac{1}{2}, \text{ so that } \theta = 60^\circ.$$

$$21. \quad \frac{1 + \tan^2 \theta}{\tan \theta} = 2 \sec \theta;$$

$$\frac{\sec^2 \theta}{\tan \theta} = 2 \sec \theta;$$

$$\therefore \sec \theta = 0, \text{ or } \frac{\sec \theta}{\tan \theta} = 2;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ.$$

$$22. \quad \frac{4}{\sin \theta} + 2 \sin \theta = 0;$$

$$2 \sin^2 \theta - 9 \sin \theta + 4 = 0;$$

$$\therefore (2 \sin \theta - 1)(\sin \theta - 4) = 0;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ.$$

$$23. \quad \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta};$$

$$\sin^2 \theta - \cos^2 \theta = \cos \theta;$$

$$1 - 2 \cos^2 \theta = \cos \theta;$$

$$2 \cos^2 \theta + \cos \theta - 1 = 0;$$

$$\therefore (2 \cos \theta - 1)(\cos \theta + 1) = 0;$$

$$\therefore \cos \theta = \frac{1}{2}, \text{ or } -1;$$

$$\therefore \theta = 60^\circ.$$

$$24. \quad 2 \cos \theta + 2\sqrt{2} = \frac{3}{\cos \theta};$$

$$2 \cos^2 \theta + 2\sqrt{2} \cos \theta - 3 = 0;$$

$$\therefore (\sqrt{2} \cos \theta - 1)(\sqrt{2} \cos \theta + 3) = 0;$$

$$\therefore \cos \theta = \frac{1}{\sqrt{2}};$$

$$\therefore \theta = 45^\circ.$$

$$25. \quad \tan \theta (2 \sin \theta - 1) = 2 \sin \theta - 1;$$

$$(2 \sin \theta - 1)(\tan \theta - 1) = 0;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ or } \tan \theta = 1;$$

$$\therefore \theta = 30^\circ \text{ or } 45^\circ.$$

$$26. \quad 6 \frac{\sin \theta}{\cos \theta} - \frac{5\sqrt{3}}{\cos \theta} + 12 \frac{\cos \theta}{\sin \theta} = 0;$$

$$6 \sin^2 \theta - 5\sqrt{3} \sin \theta + 12(1 - \sin^2 \theta) = 0;$$

$$6 \sin^2 \theta + 5\sqrt{3} \sin \theta - 12 = 0;$$

$$\therefore (2 \sin \theta - \sqrt{3})(3 \sin \theta + 4\sqrt{3}) = 0;$$

$$\therefore \sin \theta = \frac{\sqrt{3}}{2}, \text{ so that } \theta = 60^\circ.$$

$$27. \quad 5 \tan \theta + \frac{6}{\tan \theta} = 11;$$

$$5 \tan^2 \theta - 11 \tan \theta + 6 = 0;$$

$$(5 \tan \theta - 6)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } 1.2;$$

$$\therefore \theta = 45^\circ, \text{ or } 50^\circ 12'.$$

$$28. \quad 1 + \tan^2 \theta + \tan^2 \theta = 3 \tan \theta;$$

$$2 \tan^2 \theta - 3 \tan \theta + 1 = 0;$$

$$\therefore (2 \tan \theta - 1)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } \frac{1}{2}.$$

$$\therefore \theta = 45^\circ, \text{ or } 26^\circ 34'.$$

## MISCELLANEOUS EXAMPLES. A. PAGE 32.

3. If  $\theta$  be the angle, we have  $\sin \theta = \frac{21}{29}$ , so that  $\operatorname{cosec} \theta = \frac{29}{21}$ .

$$\text{Also } \cos \theta = \sqrt{1 - \left(\frac{21}{29}\right)^2} = \frac{\sqrt{(29+21)(29-21)}}{29} = \frac{20}{29}.$$

$$4. \tan A = \frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}} = \frac{15}{\sqrt{2 \times 32}} = \frac{15}{8};$$

$$\sec A = \sqrt{1 + \tan^2 A} = \frac{\sqrt{8^2 + 15^2}}{8} = \frac{17}{8}.$$

$$5. \text{First side} = \operatorname{cosec}^2 A - \cot^2 A - 1 = 0.$$

$$7. b = \sqrt{a^2 + c^2} = \sqrt{1681} = 41.$$

$$\cot A = \frac{c}{a} = \frac{9}{40}; \quad \sec A = \frac{b}{c} = \frac{41}{9}; \quad \sec C = \frac{b}{a} = \frac{41}{40}.$$

$$8. \text{See Article 16.}$$

$$9. \text{First side} = \cos \theta (1 - \cos \theta) \frac{1 + \cos \theta}{\cos \theta} = 1 - \cos^2 \theta = \sin^2 \theta.$$

$$10. \text{We have } \sec^2 a = 1 + \tan^2 a = \frac{1 + \cot^2 a}{\cot^2 a}; \quad \therefore \sec a = \frac{\sqrt{1 + \cot^2 a}}{\cot a}.$$

$$\text{Also } \operatorname{cosec}^2 a = 1 + \cot^2 a; \quad \text{so that } \operatorname{cosec} a = \sqrt{1 + \cot^2 a}.$$

$$11. \text{First side} = 3 \left( \frac{1}{\sqrt{3}} \right)^2 + \frac{1}{4} \cdot 2 + 5 \cdot 1 - \frac{2}{3} \cdot \left( \frac{\sqrt{3}}{2} \right)^2 = 1 + \frac{1}{2} + 5 - \frac{1}{2} = 6.$$

$$12. \sin a = \frac{1}{\sqrt{1 + \cot^2 a}} = \frac{m}{\sqrt{m^2 + n^2}}; \quad \sec a = \sqrt{1 + \tan^2 a} = \frac{\sqrt{m^2 + n^2}}{n}.$$

$$13. m \text{ sexagesimal minutes} = \frac{m}{60 \times 90} \text{ right angles,}$$

$$n \text{ centesimal minutes} = \frac{n}{100 \times 100} \text{ right angles,}$$

$$\therefore \frac{m}{60 \times 90} = \frac{n}{100 \times 100}; \quad \text{whence } m = 54n.$$

$$14. \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}, \text{ since } A \text{ is acute;}$$

$$\therefore \tan A + \sec A = \frac{4}{3} + \frac{5}{3} = 3.$$

$$15. \text{First side} = \tan A \cot A \sin A \cot A = \cos A.$$

$$16. RQ = \sqrt{20^2 + 21^2} = \sqrt{841} = 29;$$

$$\therefore \tan Q = \frac{RP}{PQ} = \frac{20}{21}; \quad \operatorname{cosec} Q = \frac{QR}{RP} = \frac{29}{20}.$$

$$17. \text{ First side} = \frac{\sin^2 \alpha - \cos^2 \alpha}{\sin \alpha \cos \alpha} \cdot \sin \alpha \cos \alpha = 1 - \cos^2 \alpha - \cos^2 \alpha = 1 - 2 \cos^2 \alpha.$$

$$18. \text{ See Art. 39, Ex. 1.}$$

$$19. \text{ Second side} = (\sqrt{3})^2 - 2 \cdot \left(\frac{1}{2}\right)^2 - \frac{3}{4} (\sqrt{2})^2 = 3 - \frac{1}{2} - \frac{3}{2} = 3 - 2 \\ = \tan^2 60^\circ - 2 \tan^2 45^\circ.$$

$$20. \begin{array}{ll} (1) \quad 3 \sin \theta = 2 \cos^2 \theta; & (2) \quad 5 \tan \theta - \sec^2 \theta = 3; \\ \quad 2 \sin^2 \theta + 3 \sin \theta - 2 = 0; & \quad 5 \tan \theta - 1 - \tan^2 \theta = 3; \\ \therefore (2 \sin \theta - 1)(\sin \theta + 2) = 0; & \quad \tan^2 \theta - 5 \tan \theta + 4 = 0; \\ \therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ. & \therefore (\tan \theta - 1)(\tan \theta - 4) = 0; \\ & \text{whence } \theta = 45^\circ, \text{ or } 75^\circ 58'. \end{array}$$

$$21. \text{ First side} = 1 - (\sec^2 A - \tan^2 A)^2 = 1 - 1 = 0.$$

$$22. \quad 6 \sin^2 \theta - 11 \sin \theta + 4 = 0; \quad \therefore (2 \sin \theta - 1)(3 \sin \theta - 4) = 0; \\ \therefore 2 \sin \theta - 1 = 0; \text{ whence } \theta = 30^\circ; \\ \text{or } 3 \sin \theta - 4 = 0; \text{ whence } \sin \theta = \frac{4}{3}, \text{ which is impossible.}$$

$$23. \quad \tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}.$$

$$24. \quad c + c^{-1} = \cot A + \tan A = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \sec A \operatorname{cosec} A.$$

$$25. \quad (\sin \theta + 2)(3 \sin \theta - 1) = 0, \text{ whence } \sin \theta = \frac{1}{3} = .3333; \\ \therefore \theta = 19^\circ 28'.$$

### EXAMPLES. V. a. PAGE 37.

$$1. \quad c = \sqrt{a^2 - b^2} = \sqrt{16 - 12} = 2.$$

$$\sin C = \frac{c}{a} = \frac{1}{2}; \therefore C = 30^\circ. \quad \sin B = \frac{b}{a} = \frac{\sqrt{3}}{2}; \therefore B = 60^\circ.$$

$$2. \quad a = \sqrt{b^2 - c^2} = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}.$$

$$\sin C = \frac{c}{b} = \frac{1}{2}; \therefore C = 30^\circ; \text{ and } A = 90^\circ - C = 60^\circ.$$

$$3. \quad c = \sqrt{a^2 + b^2} = \sqrt{144 + 48} = \sqrt{192} = 8\sqrt{3}.$$

$$\sin B = \frac{b}{c} = \frac{12}{8\sqrt{3}} = \frac{\sqrt{3}}{2}; \therefore B = 60^\circ; \text{ and } A = 90^\circ - B = 30^\circ.$$

$$4. \quad c = \sqrt{a^2 - b^2} = \sqrt{90 \times 30} = 30\sqrt{3}.$$

$$\sin C = \frac{c}{a} = \frac{\sqrt{3}}{2}; \therefore C = 60^\circ, B = 30^\circ.$$

$$6. \quad c = \sqrt{a^2 + b^2} = \sqrt{75 + 3 \times 75} = 10\sqrt{3}.$$

$$\sin B = \frac{b}{c} = \frac{\sqrt{3}}{2}; \quad \therefore B = 60^\circ, \quad A = 30^\circ.$$

$$7. \quad b = c = 2; \quad \therefore B = C = 45^\circ.$$

$$a = \sqrt{b^2 + c^2} = 2\sqrt{2}.$$

$$9. \quad B = 90^\circ - A = 60^\circ.$$

$$\frac{b}{a} = \tan B; \quad \therefore b = 9\sqrt{3} \cdot \sqrt{3} = 27.$$

$$\frac{c}{a} = \sec B; \quad \therefore c = 9\sqrt{3} \cdot 2 = 18\sqrt{3}.$$

$$11. \quad B = 90^\circ - A = 36^\circ.$$

$$a = c \cos B = 8 \times .8090;$$

$$b = c \sin B = 8 \times .5878.$$

$$13. \quad A = 90^\circ - C = 53^\circ.$$

$$a = b \cos C = 100 \times .7986;$$

$$c = b \sin C = 100 \times .6018.$$

$$15. \quad A = 180^\circ - B - C = 90^\circ.$$

$$b = c = 4; \quad a = \sqrt{b^2 + c^2} = 4\sqrt{2}.$$

$$17. \quad a = b \tan A = \frac{49}{.07} = 700.$$

$$19. \quad c = a \tan C = 100 \times .8647 = 86.47.$$

$$21. \quad C = 90^\circ - A = 54^\circ.$$

$$a = c \tan A = 100 \times .73 = 73;$$

$$b = c \sec A = 100 \times 1.24 = 124.$$

$$8. \quad a = \sqrt{b^2 - c^2} = \sqrt{3 \times 36 - 27} = 9.$$

$$\sin C = \frac{c}{b} = \frac{1}{2}; \quad \therefore C = 30^\circ, \quad B = 60^\circ.$$

$$10. \quad C = 90^\circ - 25^\circ = 65^\circ.$$

$$b = a \cos C = 4 \times .4226,$$

$$c = a \sin C = 4 \times .9063.$$

$$12. \quad B = 180^\circ - C - A = 90^\circ.$$

$$a = b \cos C = 6 \times .4540;$$

$$c = b \sin 63^\circ = 6 \times .8910.$$

$$14. \quad C = 180^\circ - A - B = 90^\circ.$$

$$a = b \tan A = 20; \quad c = b \sec A = 40.$$

$$16. \quad A = 180^\circ - B - C = 90^\circ.$$

$$b = a \cos C = 4; \quad c = a \sin C = 4\sqrt{3}.$$

$$18. \quad a = c \sin A = 50 \times .62 = 31.$$

$$20. \quad a = b \sec C = 200 \times 4.89 = 978.$$

$$22. \quad \sin C = \frac{c}{a} = .37; \quad \therefore C = 21^\circ 43'.$$

$$B = 90^\circ - C = 68^\circ 17';$$

$$b = a \cos C = 100 \times .93 = 93.$$

$$24. \quad c = \sqrt{a^2 + b^2} = \sqrt{124609} = 353.$$

$$23. \quad C = 90^\circ - B = 50^\circ 36'.$$

$$c = b \cot B = 25 \times 1.2174 = 30.435;$$

$$a = b \operatorname{cosec} B = 25 \times 1.5755 = 34.3875.$$

$$\tan B = \frac{b}{a} = \frac{272}{225} = 1.209;$$

$$\therefore B = 50^\circ 24'.$$

$$A = 90^\circ - B = 39^\circ 36'.$$

$$25. \quad \cos A = \frac{22.75}{25} = .91; \quad \text{whence } A = 24^\circ 30'. \quad \text{Hence } B = 65^\circ 30'.$$

$$a = c \sin A = 25 \times .4147 = 10.37, \text{ approx.}$$

**EXAMPLES. V. b. PAGE 39.**

1.  $R = 180^\circ - 30^\circ - 120^\circ = 30^\circ = A$ ;  
 $\therefore CB = CA = 20$ ;  
 $\therefore BD = BC \sin 60^\circ = 10\sqrt{3}$ .
2.  $c = BD \operatorname{cosec} 30^\circ = 20$ ;  
 $a = BD \operatorname{cosec} 45^\circ = 10\sqrt{2}$ .
3. Since  $B + C = 90^\circ$ ;  $\therefore A = 90^\circ$ .  
 $AC = AB \tan 30^\circ = 10$  ft.  $AB = BD \sec 30^\circ = 10\sqrt{3}$  ft.  
 $AD = AB \sin 30^\circ = 5\sqrt{3}$  ft.
4. Let  $QS$  be the perpendicular from  $Q$  on  $PR$ .  
Then  $PR = 8 \sec 60^\circ = 16$ .  $SR = 8 \cos 60^\circ = 4$ .  $\therefore SP = 16 - 4 = 12$
5.  $SQ = 36 \tan 53^\circ = 47.77$ .  $RQ = 36 \tan 35^\circ = 25.21$ .  
 $\therefore RS = SQ - RQ = 22.56$ .
6. We have  $\angle PRQ = 180^\circ - 135^\circ = 45^\circ$ ;  $\therefore \angle QPR = 45^\circ$ ;  
 $\therefore QR = QP = 20$ , and  $\angle QPS = 90^\circ - 25^\circ = 65^\circ$ ,  
 $\therefore SQ = 20 \tan 65^\circ = 42.89$ ,  $\therefore RS = 42.89 - 20 = 22.89$ .
7. Let  $AD$  be the perpendicular and let  $AD = x$ .  
Then  $\angle BAD = 90^\circ - 45^\circ = 45^\circ = \angle ABD$ ,  $\therefore DB = DA = x$ .  
Now  $\frac{DA}{DC} = \tan 60^\circ$ ;  $\therefore \frac{x}{x - 40} = \sqrt{3}$ ;  $\therefore x(\sqrt{3} - 1) = 40\sqrt{3}$ .  
 $\therefore x = \frac{40\sqrt{3}}{\sqrt{3} - 1} = 20\sqrt{3}(\sqrt{3} + 1) = 20(3 + \sqrt{3})$ .  
 $\therefore \text{perpendicular} = 20(3 + \sqrt{3}) = 94.64$ .
8. Let  $DC = x$ .  
Since  $\angle DCB = 90^\circ - 45^\circ = 45^\circ = \angle CBD$ ;  $\therefore DB = DC = x$ .  
And  $\frac{DA}{DC} = \cot 35^\circ 18'$ ;  $\therefore \frac{x + 41.24}{x} = 1.4124$ .  
 $\therefore x + 41.24 = 1.4124x$ ;  $\therefore x = 100$ ;  
that is,  $DC = DB = 100$ .
9. The perp.  $AD = 20 \sin 42^\circ = 20 \times .6691 = 13.382$ ,  
 $\tan C = \frac{AD}{CD} = \frac{13.382}{18.138} = .7378$ ;  
whence  $C = 36^\circ 25'$ .

**EXAMPLES. VI. a. PAGE 42.**

For Examples 1—5 see figure on page 40.

1. Let  $BC$  = height of chimney,  $AC = 300$  ft.,  
then elevation =  $\angle BAC = 30^\circ$ ,  $\therefore BC = AC \tan 30^\circ = 100\sqrt{3} = 173.2$  ft.

2. Let  $B$  be the top of the mast,  $BC = 160$  feet,  $A$  the boat observed.  
 Then  $\angle BAC = 30^\circ$ ,  $\angle ABC = 60^\circ$ .  
 $\therefore$  distance required  $= AC = BC \tan 60^\circ = 160\sqrt{3} = 277.12$  ft.

3. Let  $BC$  represent the pole, and  $AC$  its shadow.

Then  $\tan A = \frac{BC}{AC} = \frac{6}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ ;  $\therefore$  angle of elevation  $= 60^\circ$ .

4. Let  $BC$  represent the tower;  $A$  the position of the observer.  
 Then  $AC = 86.6$  ft. and  $\angle BAC = 30^\circ$ .

$\therefore$  height of tower  $= AC \tan 30^\circ = \frac{86.6}{\sqrt{3}} = 50$  ft.

Distance  $AB = BC \operatorname{cosec} 30^\circ = 2BC = 100$  ft.

5. Let  $AB$  represent the ladder, and  $BC$  the wall.

Then  $AB = 45$  ft.  $\angle ABC = 60^\circ$ .

$\therefore$  height of wall  $= BC = AB \cos 60^\circ = 22.5$  ft.

Distance  $AC = AB \sin 60^\circ = \frac{45\sqrt{3}}{2} = 38.97$  ft.

6. See figure on page 9.

Let  $DE$ ,  $BC$  represent the masts, and  $OCE$  the horizon.

Then  $\angle BOC = 33^\circ 41'$ .  $BC = 40$  ft.  $DE = 60$  ft.

And  $OC = BC \cot 33^\circ 41' = 60$  ft.  $OE = DE \cot 33^\circ 41' = 90$  ft.

$\therefore$  distance required  $= OE - OC = 30$  ft.

7. See figure on page 40.

Let  $BC$  represent the cliff and  $A$  the observer.

Then  $\angle BAC = 41^\circ 18'$ .  $BC = 132$  yds.

$\therefore$  distance required  $= AB = BC \operatorname{cosec} 41^\circ 18' = \frac{132}{\sin 41^\circ 18'} = 200$  yds.

8. See figure on page 9.

Let  $BC$ ,  $DE$  represent the chimneys, and  $O$  the observer.

Then  $OC = 100$  yds.  $\angle BOC = 27^\circ 2'$ .

Now  $BC = OC \tan 27^\circ 2' = 51$  yds.

$\therefore DE = BC + 30$  yds.  $= 81$  yds.

9. See figure on page 41.

Let  $PT$  be the tower, and  $Q$ ,  $R$  the two points of observation.

Then  $\angle PQR = 30^\circ$ ,  $\angle PRT = 60^\circ$ ,  $QR = 100$  yds.

$\therefore \angle RPQ = 60^\circ - 30^\circ = 30^\circ = \angle PQR$ ,  $\therefore RP = RQ = 100$  yds.

$\therefore$  height of tower  $= RP \sin 60^\circ = 50\sqrt{3} = 86.6$  yds.

10. Let  $AB$  be the flagstaff,  $BC$  the building,  $D$  the point of observation.  
 Then  $DC = 40$  ft.  $\angle ADC = 60^\circ$ ,  $\angle BDC = 30^\circ$ ;  
 $\therefore \angle DAB = 30^\circ = \angle ADB$ ;  $\therefore BA = BD = 40 \sec 30^\circ = \frac{80}{\sqrt{3}} = 49.19$  ft.

11. See figure on page 41.

Let  $PT$  be the spire,  $R, Q$  the two points of observation.

Then  $QR = 200$  ft.  $\angle PRT = 45^\circ$ ,  $\angle PQR = 30^\circ$ .  
 $\angle RPT = 45^\circ$ ;  $\therefore TP = TR$ .

Let  $x$  ft. = height of spire.

Then  $\frac{x}{x+200} = \tan 30^\circ = \frac{1}{\sqrt{3}}$ .

$\therefore x = \frac{200}{\sqrt{3}-1} = 100(1+\sqrt{3}) = 273.2$  feet.

12. See figure on page 43.

Let  $CD$  represent the post, and  $AB$  the steeple.

Then  $CD = 30$  ft.  $\angle ACE = 30^\circ$ ,  $\angle ADB = 45^\circ$ ;

$\therefore \angle DAB = 45^\circ = \angle ADB$ ,  $\therefore BA = BD = x$  feet, say.

$\therefore \tan 30^\circ = \frac{AE}{EC} = \frac{x-30}{x}$ ;  $\therefore x(\sqrt{3}-1) = 30\sqrt{3}$ .  $\therefore x = 70.98$  ft.

That is height = distance = 70.98 ft.

13. Let  $B$  be the top of the hill, and  $C$  the point on the horizontal plane vertically below  $B$ . Let  $D$  be the position of the balloon when the observation is made. Draw  $DE$  perpendicular to  $BC$ .

Then  $BC = 3300$  ft.  $\angle BDE = 30^\circ$ ,  $\angle BAC = 60^\circ$ ;

$\therefore AC = BC \cot 60^\circ = 1100\sqrt{3}$  feet.

And  $BE = DE \tan 30^\circ = AC \tan 30^\circ = 1100$  feet.

$\therefore DA = EC = 3300 - 1100 = 2200$  feet.

$\therefore$  the balloon rises 2200 feet in 5 minutes,

that is,  $\frac{2200 \times 60}{1760 \times 3 \times 5}$  miles per hour, or 5 miles per hour.

14. See figure on page 44.

Let  $OA$  represent the monument,  $B, C$  the two objects,  $OP$  the horizontal line through  $O$ ;

Then  $\angle POC = 30^\circ$ ;  $\therefore \angle OCB = 30^\circ$ .

$\angle POB = 45^\circ$ ;  $\therefore \angle BOA = \angle OBA = 45^\circ$ ;  $\therefore AO = AB = 100$  feet.

Let  $x$  feet =  $CB$  = required distance.

Then  $\frac{100}{x+100} = \tan 30^\circ = \frac{1}{\sqrt{3}}$ ;  $\therefore x = 100(\sqrt{3}-1)$ ;

$\therefore$  distance required = 73.2 feet.

5. See figure on page 43.

Let  $AB$  represent the monument,  $CD$  the tower.

Then  $AB = 96$  feet, and the angles are as in the figure;

$$\therefore DB = AB \cot 60^\circ = 32\sqrt{3} \text{ feet};$$

$$\therefore AE = CE \tan 30^\circ = DB \tan 30^\circ = 32 \text{ feet.}$$

$$\therefore \text{height of tower} = CD = EB = 96 - 32 = 64 \text{ feet.}$$

16. See figure on page 44.

Let  $OA$  represent the cliff,  $B, C$  the two boats.

Then  $OA = 150$  ft.,  $\angle OBA = 30^\circ$ ,  $\angle OCB = 15^\circ$ ;

$$\therefore \angle BOC = 15^\circ, \text{ and } BC = BO.$$

$$\therefore \text{required distance} = BC = BO = AO \operatorname{cosec} 30^\circ = 300 \text{ ft.}$$

17. See figure on page 44.

Let  $O$  represent the top of the hill,  $B, C$  the milestones.

Then  $\angle OBA = 45^\circ$ ,  $\angle OCB = 22^\circ$ ;  $\therefore \angle BOA = 45^\circ$ , so that  $AO = AB$ .

Let  $AO = AB = x$  yards.

$$\text{Then } \frac{AC}{AO} = \frac{x + 1760}{x} = \cot 22^\circ = 2.475; \therefore 1.475x = 1760.$$

$$\therefore \text{height of hill} = x = 1193 \text{ yds. nearly.}$$

18. See figure on page 44.

Let  $OA$  represent the lighthouse,  $B, C$  the two rocks.

Then  $OA = 80$  yds.,  $\angle OBA = 75^\circ$ ,  $\angle OCB = 15^\circ$ ,  $\angle COA = 75^\circ$ ;

$$\therefore AB = OA \cot 75^\circ = 80 \times .268 \text{ yds.}$$

Let  $CB = x$  yds.

$$\text{Then } x + 80 \times .268 = OA \cot 15^\circ = 80 \times 3.732, \therefore x = 80 \times 3.464 = 277.12 \text{ yds.}$$

$$\therefore \text{required distance} = 277.12 \text{ yds.}$$

### EXAMPLES. VI b. PAGE 47.

1. Let  $A, B$  be the two positions of the observer,  $P, Q$  the two objects. Then  $AB = 800$  yds., and  $PQA$  is a straight line making  $\angle PAB$  equal to  $45^\circ$ . Also  $\angle PBA = 90^\circ$ ,  $\angle QBA = 45^\circ$ .  $\therefore QA = QB = QP$ .

$$\text{And } QA = AB \cos 45^\circ = \frac{800}{\sqrt{2}} = 565.6 \text{ yds. } PA = 2QA = 1131.2 \text{ yds.}$$

Thus the required distances are 565.6 yds., 1131.2 yds.

2. Let  $A, B$  be the two positions of the observer,  $P, Q$  the two ships; then  $APQ$  is a straight line at right angles to  $AB$ .

And  $AB = 3$  miles,  $\angle ABP = 30^\circ$ ,  $\angle ABQ = 60^\circ$ .

$$\therefore BP = AB \sec 30^\circ = \frac{6}{\sqrt{3}} = 3.464 \text{ miles, } BQ = AB \sec 60^\circ = 6 \text{ miles.}$$

Thus the required distances are 3.464 miles, 6 miles.

3. Let  $O$  represent the harbour and  $ON, OE, OS, OW$  the directions of North, South, East, West.

Let  $P, Q$  be the positions of the two ships at 2 p.m.

Then  $\angle POW = 28^\circ, \angle QOE = 62^\circ; \therefore \angle POQ = 90^\circ$ .

Also  $OP = 2 \times 10 = 20$  miles.  $OQ = 2 \times 10\frac{1}{2} = 21$  miles.

$$\therefore \text{distance} = PQ = \sqrt{20^2 + 21^2} = 29 \text{ miles.}$$

4. Let  $O$  be the position of the lighthouse, and  $P, Q$  the points at which the steamer enters and leaves the light.

Then  $PQ$  lies East and West, and  $OP, OQ$  are the directions of N.E., N.W.

$$\therefore \angle POQ = 90^\circ, \angle PQO = \angle QPO = 45^\circ, OP = OQ = 5 \text{ miles.}$$

$$\therefore PQ = \sqrt{25 + 25} = 5\sqrt{2} \text{ miles. } \therefore \text{steamer sails } 5\sqrt{2} \text{ miles in } 30\sqrt{2} \text{ minutes,}$$

that is, the speed of steamer is 10 miles per hour.

5. Let  $O, P, Q$  be the first positions of the ship and lighthouses.  $OA$  the direction in which the ship is sailing,  $A$  its second position.

$$\text{Then } OA = 10 \text{ miles, } \angle OAP = 45^\circ, \angle AOP = 90^\circ, \angle PAQ = 22\frac{1}{2}^\circ.$$

$$\therefore \angle PQA = 45^\circ - 22\frac{1}{2}^\circ = 22\frac{1}{2}^\circ = PAQ; \therefore PA = PQ.$$

$$\text{And } OP = OA = 10 \text{ miles, } PA = OA \sec 45^\circ = 10\sqrt{2} \text{ miles.}$$

$$\therefore OQ = OP + PQ = OP + PA = 10(\sqrt{2} + 1) = 24.14 \text{ miles.}$$

$$\therefore \text{distances are } 10 \text{ miles, } 24.14 \text{ miles.}$$

6. As before let  $O$  be the port, and  $ON, OE, OS, OW$  the directions of the cardinal points of the compass.

Let  $P, Q$  be the positions of the ships at the end of an hour.

$$\text{Then } OP = 8 \text{ miles, } OQ = 8\sqrt{3} \text{ miles.}$$

$$\text{And } \angle PON = 35^\circ, \angle QOS = 55^\circ; \therefore \angle POQ = 90^\circ.$$

$$\therefore \text{distances apart} = PQ = \sqrt{64 + 3 \times 64} = 16 \text{ miles.}$$

$$\text{Also } \tan QPO = \frac{8\sqrt{3}}{8} = \sqrt{3}; \therefore \angle QPO = 60^\circ.$$

$$\therefore \angle QPO - \angle PON = 60^\circ - 35^\circ = 25^\circ;$$

$\therefore$  bearing of the second vessel as observed from the first is S.  $25^\circ$  W.

7. Let  $A$  be the lighthouse,  $O, P$  the two positions of the vessel.

Then  $AP$  the direction of S.,  $AO$  the direction of E.S.E.,  $OP$  the direction of S.S.W.

$$\therefore \angle AOP = 90^\circ; \angle PAO = 90^\circ - 22\frac{1}{2}^\circ = 67\frac{1}{2}^\circ; \text{ and } AO = 4 \text{ miles.}$$

$$\therefore PO = AO \tan 67\frac{1}{2}^\circ = 4 \times 2.414 = 9.656 \text{ miles.}$$

$$\therefore \text{the vessel sails at the rate of } 9.656 \text{ miles per hour.}$$

8. We have  $\angle CAB = 10^\circ + 50^\circ = 60^\circ$ ,  $\angle ABC = 180^\circ - 50^\circ - 40^\circ = 90^\circ$ ,

$$BC = 10 \text{ miles.}$$

$$\therefore AB = BC \cot 60^\circ = \frac{10}{\sqrt{3}} = 5.77 \text{ mls.}; AC = BC \operatorname{cosec} 60^\circ = \frac{20}{\sqrt{3}} = 11.54 \text{ mls.}$$

9. Let  $O$  be the lighthouse, and  $A, B$  the two positions of the ship.

Then  $\angle OAB = \angle OBA = 45^\circ$ ;  $OA = OB = 15$  miles.

$$\therefore AB = 15\sqrt{2} \text{ miles} = \frac{15\sqrt{2} \times 60}{69} \text{ knots.}$$

$$\therefore \text{the ship in } 1\frac{1}{2} \text{ hours sails } \frac{15\sqrt{2} \times 60}{69} \text{ knots.}$$

$$\therefore \text{in a day it sails } \frac{15\sqrt{2} \times 60}{69} \times \frac{24 \times 2}{3} \text{ knots, that is, } 295.09 \text{ knots.}$$

10. Let  $O$  be the lighthouse,  $A, B$  the two positions of the coaster.  
Then  $AB$  is in direction S.E., and  $OA$  is in direction N.E.;  $\therefore \angle OAB = 90^\circ$ .

Also  $\angle AOB = 45^\circ + 15^\circ = 60^\circ$ , and  $OA = 9$  miles;

$$\therefore AB = OA \tan 60^\circ = 9\sqrt{3} \text{ miles}; \therefore \text{coaster sails } 9\sqrt{3} \text{ miles in 3 hours.}$$

$$\therefore \text{rate of the coaster's sailing} = 5.196 \text{ miles per hour.}$$

$$\text{Also } OB = AO \sec 60^\circ = 18 \text{ miles.}$$

That is, the distance of the coaster from the lighthouse at time of second observation = 18 miles.

11. Let  $P$  be the position of the vessel when it is N.E. of  $A$  and N.W. of  $B$ . Then  $\angle APB = 90^\circ$ .

$$\text{Also } \angle PAB = 45^\circ - 15^\circ = 30^\circ; \therefore PA = AB \cos 30^\circ = 6\sqrt{3} \text{ miles.}$$

Now the direction S.  $15^\circ$  E. is at right angles to the direction E.  $15^\circ$  N.

$\therefore$  the ship crosses  $AB$  at right angles. Draw  $PN$  perpendicular to  $AB$ .

Then  $PN = AP \sin 30^\circ = 3\sqrt{3}$  miles; therefore the ship will reach  $N$  in

$$\frac{3\sqrt{3}}{10} \text{ hours, that is in } 31.176 \text{ minutes.}$$

$$\therefore \text{the ship will cross the line at about } 31' \text{ past midnight.}$$

12. Let  $P, Q$  be the two spires.

Then  $\angle PAB = 90^\circ$ , and  $\angle PBQ = 37\frac{1}{2}^\circ - 7\frac{1}{2}^\circ = 30^\circ$ ;

$$\therefore \angle QBA = 90^\circ - 37\frac{1}{2}^\circ - 22\frac{1}{2}^\circ = 30^\circ;$$

$$\therefore \angle BPQ = 30^\circ = \angle PBQ; \text{ so that } QB = QP = 1.5 \text{ miles.}$$

$$\therefore AB = BQ \cos 30^\circ = \frac{3\sqrt{3}}{4} \text{ miles.}$$

$$\therefore \text{the train travels } \frac{3\sqrt{3}}{4} \text{ miles in 2 minutes,}$$

$$\text{that is, } \frac{3\sqrt{3}}{4} \times 30 \text{ miles per hour, or } 38.97 \text{ miles per hour.}$$

## EXAMPLES. VI. c. PAGE 48 A.

$$1. \quad h = 83 \tan 23^\circ 44' = 83 \times .4397 \text{ yards} \\ = 109 \text{ ft., approximately.}$$

$$2. \quad h = 173 \tan 63^\circ = 173 \times 1.9626 \text{ ft.} \\ = 339.53 \text{ ft.}$$

$$3. \quad h = 200 \sin 54^\circ = 200 \times .8090 \text{ metres} \\ = 161.8 \text{ metres.}$$

$$4. \quad d = 500 \sin 23^\circ = 500 \times 3 \times .3907 \text{ ft.} \\ = 586.05 \text{ ft.}$$

$$5. \quad \text{The distance between two consecutive posts} = \frac{1760}{22} = 80 \text{ yds.}$$

$$\text{Then required distance} = 80 \tan 16^\circ 42' = 80 \times .3000 \text{ yds.} \\ = 24 \text{ yds.}$$

6. Let  $ABCD$  be the square, and let the line be drawn from  $B$  to  $E$ , the middle point of  $AD$ .

$$\text{Then} \quad \tan ABE = \frac{AE}{AB} = .5, \text{ whence } \angle ABE = 26^\circ 34'; \\ \therefore \angle EBC = 90^\circ - 26^\circ 34' = 63^\circ 26'.$$

7. Let  $D$  be the middle point of the base  $BC$  of the isosceles  $\triangle ABC$ , in which  $AB = 3BC$ .

$$\text{Then} \quad \cos DBA = \frac{BD}{BA} = \frac{1}{6} = .1666,$$

$$\text{whence} \quad \angle B = 80^\circ 25' = \angle C; \quad \therefore A = 19^\circ 10'.$$

8. With the figure on p. 41,  $QR = 160 \text{ ft.}$ ,  $\angle PRT = 45^\circ$ ,  $\angle PQT = 21^\circ 48'$ ; also  $PT = RT$ . If  $h$  is the required height in feet

$$\frac{h}{h + 160} = \tan 21^\circ 48' = .4000;$$

$$\therefore h = .4h + 64; \quad \therefore .6h = 64, \text{ and } h = 107.$$

9. With the same figure as in Ex. 8,  $QR = 100 \text{ yds.}$ ,  $\angle PRT = 54^\circ 24'$ ,  $\angle PQT = 27^\circ 12'$ . Let  $h$  be the height in feet; then

$$\frac{300 + RT}{h} = \cot 27^\circ 12' = \tan 62^\circ 48' = 1.9458;$$

$$\text{and} \quad RT = h \cot 54^\circ 24' = h \tan 35^\circ 36' = h \times .7159; \\ \therefore 300 + h \times .7159 = h \times 1.9458;$$

that is,  $1.2299h = 300$ ; whence  $h = 244$ , nearly.

**Or thus:** Since  $\angle RPQ = 27^\circ 12'$ ,  $\therefore PR = QR = 100 \text{ yds.}$

$$\therefore h = 300 \sin 54^\circ 24' = 300 \times .8131 = 244.$$

## EASY PROBLEMS.

VI.]

10. With the same figure and notation as in Ex. 9,

$$\frac{1760 + RT}{h} = \tan 73^\circ 18' = 3.3332;$$

$$RT = h \tan 35^\circ = h \times .7002;$$

$$\therefore 1760 + h \times .7002 = h \times 3.3332;$$

that is,  $2.6330h = 1760$ ; whence  $h = 668$ , nearly.

11. Let  $AB$  represent the top, and  $DE$  the bottom of the trench. Draw  $AC$  and  $BF$  perp. to  $ED$  and  $DE$  produced.

Then

$$CD = 8 \tan 12^\circ = 8 \times .2126 = 1.7008 \text{ ft.};$$

$$CE = 10.7008 \text{ ft.} \therefore EF = 4.2992 \text{ ft.}$$

whence

$$\text{In } \triangle EBF, \quad \tan B = \frac{4.2992}{8} = .5374;$$

$$\text{whence } B = 28^\circ 15'.$$

12. Let  $C$  and  $B$  be the first and second positions of the observer; then  $\angle ACB = 90^\circ$ , and  $\angle ADB = 143^\circ 24'$ .  $\therefore \angle ADC = 36^\circ 36'$ .

$$\text{Now} \quad AC = 630 \tan 36^\circ 36' = 630 \times .7427 = 467.9 \text{ m.}$$

$$AD = \frac{63}{\cos 36^\circ 36'} = \frac{63}{.8028} = 784.7 \text{ m.}$$

13. Let  $A$  be the point of observation,  $B$  the top, and  $C$  the bottom of the tower. Draw  $AE$  horizontally to meet  $BC$  in  $E$ . Then  $\angle EAC = 17^\circ$ , and  $EC = AD = 30 \text{ ft.}$

$$AE = EC \cot 17^\circ = 30 \tan 73^\circ = (30 \times 3.2709) \text{ ft.} \\ = 98.127 \text{ ft.}$$

Again

$$BE = AE \tan 42^\circ = (98.127 \times .9004) \text{ ft.} \\ = 88.3506 \text{ ft.};$$

$$\therefore \text{required height} = (30 + 88.3506) \text{ ft.} \\ = 118.35 \text{ ft.}$$

14. See fig. on page 44. Let  $OA = x$ ,  $AB = y$ ;  
then

$$y = x \cot 35^\circ = x \tan 55^\circ = x \times 1.4281,$$

$$\frac{700 + y}{x} = \cot 14^\circ = \tan 76^\circ = 4.0108;$$

$$\therefore 700 + x \times 1.4281 = x \times 4.0108;$$

$$700 = x \times 2.5827; \text{ whence } x = 271.$$

that is,

15. Let  $O$  be the lighthouse, and  $A, B$  the two positions of the ship. Then  $\angle BOA = 90^\circ$ ,  $\angle OBA = 32^\circ$ ,  $AB = 15 \text{ mi.}$

Now

$$OA = AB \sin ABO = 15 \sin 32^\circ \\ = 15 \times .5299 = 7.9485 \text{ mi.}$$

16. Let  $O$  be the house, and  $A, B$  the two positions. Then  $\angle BOA = 90^\circ$ ,  $\angle OBA = 52^\circ$ ,  $AB = 2$  km. Let  $OC$  be perp. to  $AB$ ;

then  $OB = 2000 \cos 52^\circ = 2000 \times .6157 = 1231.4$  m.

$$OC = OB \sin 52^\circ = 1231.4 \times .7880 = 970.3 \text{ m.}$$

17. Let  $L$  be the lighthouse, and  $S_1, S_2$  the two positions of the ship. Then  $\angle S_1OS_2 = 90^\circ$ ,  $\angle S_2S_1L = 56^\circ$ ,  $LS_1 = 12$  mi.

Now  $S_1S_2 = \frac{12}{\cos 56^\circ} = \frac{12}{.5592}$  mi.; and since the ship has been sailing  $\frac{7}{6}$

of an hour, the number of miles per day  $= \frac{12}{.5592} \times \frac{6}{7} \times 24 = 441.5$ .

18. Let  $B$  be the battery, and  $S_1, S_2$  the two positions of the ship. Draw  $S_1C$  perp. to  $BS_2$ ; then  $\angle CS_1B = 45^\circ$ ,  $S_1B = 2.5$  mi., and  $BS_2 = 4$  mi.

Now  $S_1C = 2.5 \sin 45^\circ = 2.5 \times .7071 = 1.7678$  mi.

$$S_2C = 4 - 1.7678 = 2.2322 \text{ mi.}$$

$$\tan S_1S_2C = \frac{1.7678}{2.2322} = .7912; \text{ whence } \angle S_1S_2C = 38^\circ 23'.$$

$$\therefore S_2 \text{ lies } 38^\circ 23' \text{ E. of N. from } S_1.$$

19. See fig. on page 47.

Here  $\angle BAE = 49^\circ$ ,  $\angle EAC = 41^\circ$ ;  $\therefore \angle BAC = 90^\circ$ .

Also  $\angle ACN' = 90^\circ - 41^\circ = 49^\circ$ , and  $\angle BCN' = 15^\circ$ ;

$$\therefore \angle ACB = 49^\circ - 15^\circ = 34^\circ.$$

Now  $AB = AC \tan 34^\circ = 20 \times .6745 = 13.49$  mi.

$$BC = \frac{AC}{\cos 34^\circ} = \frac{20}{.8290} = 24.12 \text{ mi.}$$

### EXAMPLES. VII. a. PAGE 54.

For Examples 1—22, see Art. 64; the following solutions will suffice as illustrations.

6. Radian measure of  $57\frac{1}{2}$  degrees  $= \frac{57\frac{1}{2}}{180} \pi = \frac{23\pi}{72}$ .

7. Radian measure of  $14\frac{2}{3}$  degrees  $= \frac{14\frac{2}{3}}{180} \pi = \frac{2\pi}{25}$ .

10. Radian measure of  $37\frac{1}{2}$  degrees  $= \frac{37\frac{1}{2}}{180} \times 3.1416 = .6545$ .

$$11. \text{ Radian measure of } 68\frac{3}{4} \text{ degrees} = \frac{68\frac{3}{4}}{180} \times 3.1416$$

$$= \frac{275}{4 \times 180} \times 3.1416 = \frac{55}{4 \times 3} \times .2618 = 1.1999.$$

$$16. \frac{7\pi}{45} \text{ radians} = \frac{7 \times 180}{45} \text{ degrees} = 28^\circ.$$

$$19. .3927 \text{ radians} = .3927 \times \frac{180^\circ}{\pi} \text{ [Art. 63]}$$

$$= \frac{3927}{31416} \times 180^\circ = \frac{180^\circ}{8} = 22^\circ 30'.$$

$$22. 2.8798 \text{ radians} = 2.8798 \times \frac{180^\circ}{\pi} = \frac{28798}{31416} \times 180^\circ = \frac{11}{12} \times 180^\circ = 165^\circ.$$

$$23. \text{ Here } \frac{\theta}{\pi} = \frac{36.54}{180} = \frac{2.03}{10};$$

$$\begin{array}{r} 60 \overline{) 24} \\ 60 \overline{) 324} \\ \underline{54} \end{array}$$

$$\therefore \theta = .203 \times \frac{22}{7} = .638.$$

$$25. \text{ Here } \frac{\theta}{\pi} = \frac{116.046}{180} = .6447;$$

$$\begin{array}{r} 60 \overline{) 456} \\ 60 \overline{) 276} \\ \underline{046} \end{array}$$

$$\therefore \theta = .6447 \times \frac{22}{7} = 2.0262.$$

$$27. \text{ A radian} = \frac{180}{\pi} \text{ degrees};$$

$$\begin{aligned} \therefore \text{no. of seconds in a radian} &= \frac{180 \times 60 \times 60}{\pi} \\ &= 180 \times 60 \times 60 \times .31831 \\ &= 206265 \text{ nearly.} \end{aligned}$$

$$28. \text{ Since } 1^\circ = \frac{\pi}{180} \text{ radians, the radian measure of } 1''$$

$$= \frac{3.1416}{180 \times 60 \times 60} = .0000048.$$

### EXAMPLES. VII. b. PAGE 56.

$$5. \cot^2 \frac{\pi}{6} + 4 \cos^2 \frac{\pi}{4} + 3 \sec^2 \frac{\pi}{6} = (\sqrt{3})^2 + 4 \left( \frac{1}{\sqrt{2}} \right)^2 + 3 \left( \frac{2}{\sqrt{3}} \right)^2 = 3 + 2 + 4 = 9.$$

$$\begin{aligned} 6. \quad 3 \tan^2 \frac{\pi}{6} - \frac{1}{3} \sin^2 \frac{\pi}{3} - \frac{1}{2} \operatorname{cosec}^2 \frac{\pi}{4} + \frac{4}{3} \cos^2 \frac{\pi}{6} \\ = 3 \left( \frac{1}{\sqrt{3}} \right)^2 - \frac{1}{3} \left( \frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} (\sqrt{2})^2 + \frac{4}{3} \left( \frac{\sqrt{3}}{2} \right)^2 \\ = 1 - \frac{1}{4} - 1 + 1 = \frac{3}{4}. \end{aligned}$$

$$7. \left( \sin \frac{\pi}{6} + \cos \frac{\pi}{6} \right) \left( \sin \frac{\pi}{3} - \cos \frac{\pi}{3} \right) \sec \frac{\pi}{3} \\ = \left( \frac{1}{2} - \frac{\sqrt{3}}{2} \right) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) 2 = 2 \left( \frac{3}{4} - \frac{1}{4} \right) = 1.$$

$$8. \text{ First side} = \sin \theta \operatorname{cosec} \theta - \cot \theta \tan \theta = 1 - 1 = 0.$$

$$9. \text{ First side} = \frac{\cos^2 \theta}{\sin \theta} - \cot^2 \theta \sin \theta = \frac{\cos^2 \theta}{\sin \theta} - \frac{\cos^2 \theta}{\sin \theta} = 0.$$

$$10. \text{ First side} = \frac{\cos^2 \theta}{\operatorname{cosec} \theta} \cdot \frac{\sec \theta}{\tan \theta} = \cos^2 \theta \sin \theta \times \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} = \cos^2 \theta.$$

$$11. \text{ First side} = \tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ = \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} = \sec \theta \operatorname{cosec} \theta = \sec \theta \sec \left( \frac{\pi}{2} - \theta \right).$$

$$12. \text{ First side} = \sec^2 \theta + \operatorname{cosec}^2 \theta = \sec^2 \theta (1 + \cot^2 \theta) \\ = \sec^2 \theta \operatorname{cosec}^2 \theta = (1 + \tan^2 \theta) \sec^2 \left( \frac{\pi}{2} - \theta \right).$$

$$14. \text{ See Art. 69.}$$

$$15. \text{ The second side} = \frac{1}{\cos^2 \frac{\pi}{3}} - \frac{1}{\cos^2 \frac{\pi}{6}} = 4 - \frac{4}{3} = \frac{8}{3} \\ = 3 - \frac{1}{3} = \tan^2 \frac{\pi}{3} - \cot^2 \frac{\pi}{3}.$$

$$16. \text{ The expression} = (\sin \theta + \cos \theta)^2 + (\cos \theta - \sin \theta)^2 \\ = 2(\sin^2 \theta + \cos^2 \theta) = 2.$$

### EXAMPLES. II. c. PAGE 60.

$$1. \text{ Here } \frac{\text{arc}}{\text{radius}} = \frac{1.6}{8} = \frac{1}{5} = \text{radian measure required.}$$

$$2. \text{ Here } r = \frac{a}{\theta} = \frac{219}{.73} = 300 \text{ ft.}$$

$$3. \text{ Radian measure} = \frac{7.5}{7.5} = 1.$$

$$4. \text{ Here } a = r\theta = 1.625 \times 3.6 = 5.85 \text{ yds.}$$

$$5. \text{ Here } a = 627 \text{ inches; } \theta = 1.9. \therefore r = \frac{a}{\theta} = \frac{627}{1.9} = 330 \text{ inches.}$$

$$6. \text{ Each revolution} = 2\pi \text{ radians;}$$

$$\therefore 5 \text{ radians} = \frac{5}{2\pi} = \frac{7 \times 5}{44} \text{ revolutions.}$$

And each revolution takes  $\frac{1}{35}$  of a second;

$$\therefore \text{required time} = \frac{1}{35} \times \frac{35}{44} = \frac{1}{44} \text{ of a second.}$$

7. Here  $a = r\theta$ , where  $r = 28$  inches and  $\theta = \frac{1}{3} \times \frac{44}{7}$ .

$$\therefore a = \frac{44}{21} \times 28 = 58\frac{2}{3} \text{ inches.}$$

8. Radian measure of  $75^\circ = \frac{75 \times 3.1416}{180}$ .

If  $r$  be the length of the rope in yards, we have

$$r = \frac{52.36 \times 180}{3.1416 \times 75} = 40 \text{ yards.}$$

9. Here  $a = r\theta$ , where  $r = 3960$  miles, and  $\theta =$  radian measure of 1 minute;

$$\therefore a = 3960 \times \frac{3.1416}{180 \times 60} = 1.15192 \text{ miles.}$$

10. The number of radians in the angle  $= \frac{11}{2} \times \frac{1}{1760 \times 12 \times 3}$ ;

$$\therefore \text{number of seconds} = \frac{11}{2} \times \frac{1}{1760 \times 12 \times 3} \times \frac{180}{3.1416} \times 60 \times 60 \\ = 17.904, \text{ on reduction.}$$

11. With the figure on p. 59, we have to find the angle  $POQ$  when  $PO = 3960$  miles, and  $PQ = 145.2$ .

$$\therefore \text{radian measure} = \frac{145.2}{3960};$$

$$\therefore \text{no. of degrees} = \frac{145.2}{3960} \times \frac{180 \times 7}{22} = \frac{1452}{220} \times \frac{7}{22} = \frac{66 \times 7}{220} = \frac{21}{10}.$$

$$\therefore \text{the angle} = 2^\circ 6'.$$

12. Here  $r = \frac{a}{\theta}$ , where  $a = 1$  foot, and  $\theta$  is the radian measure of  $\frac{11}{11}$  degrees.

$$\therefore r = 1 \div \left( \frac{14}{11} \times \frac{\pi}{180} \right) = \frac{11}{14} \times \frac{180 \times 7}{22} = 45 \text{ ft.}$$

### MISCELLANEOUS EXAMPLES. B. PAGE 61.

1.  $D = \frac{180}{\pi} \times .15708 = \frac{180 \times 15708}{314160} = 9.$

2. See figure on p. 14.  $b = c \cos A = \frac{110\sqrt{3}}{2} = 55 \times 1.732 = 95.26.$

3. If  $\frac{12\pi}{23}$  is represented by  $\frac{8}{5}$ ,

$$\pi \dots\dots\dots \frac{8}{5} \times \frac{25}{12}, \text{ or } \frac{10}{3}.$$

$\therefore 180 \div \frac{10}{3}$  is the number of degrees in the unit angle; that is, the unit is  $54^\circ$ .

4. Here  $r = \frac{a}{\theta}$ , where  $a = 1$  inch and  $\theta$  is the radian measure of  $1'$ .

$$\therefore r = 1 \div \left( \frac{\pi}{180} \times \frac{1}{60} \right) = 180 \times 60 \times \frac{1}{\pi} = 180 \times 60 \times .31831 = 3438 \text{ inches.}$$

$$\begin{aligned} 5. (1) \text{ First side} &= (\sin a + \cos a) \left( \frac{\sin a}{\cos a} + \frac{\cos a}{\sin a} \right) \\ &= \frac{(\sin a + \cos a)(\sin^2 a + \cos^2 a)}{\sin a \cos a} = \sec a + \operatorname{cosec} a. \end{aligned}$$

$$\begin{aligned} (2) \text{ First side} &= (\sqrt{3} + 1)(3 - \sqrt{3}) = \sqrt{3}(\sqrt{3} + 1)(\sqrt{3} - 1) \\ &= 3\sqrt{3} - \sqrt{3} = \tan^3 60^\circ - 2 \sin 60^\circ. \end{aligned}$$

6. In the figure on p. 14, if  $BC$  represents the chimney and  $AO$  the shadow we have

$$\tan A = \frac{60}{3 \times 20\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}; \quad \therefore A = 30^\circ.$$

$$\begin{aligned} 7. (1) \text{ First side} &= 2 \tan^2 \theta + 5 \tan \theta + 2 \\ &= 2(1 + \tan^2 \theta) + 5 \tan \theta = 2 \sec^2 \theta + 5 \tan \theta. \end{aligned}$$

$$(2) \frac{\cot^2 a}{1 + \operatorname{cosec} a} = \frac{\operatorname{cosec}^2 a - 1}{\operatorname{cosec} a + 1} = \operatorname{cosec} a - 1.$$

8. Expressed in radians the third angle  $= \pi - \left( \frac{\pi}{4} + \frac{5\pi}{8} \right) = \frac{\pi}{8}$ . The sexagesimal equivalent is  $22\frac{1}{2}^\circ$ .

9. Let  $x$  be the number of degrees in the angle; then

$$x = 14 \left( \frac{\pi}{180} x \right) + 51, \text{ or } x - \frac{11}{45} x = 51; \text{ whence } x = 67\frac{1}{2}.$$

10. With the figure of Art. 45, we have

$$a = b \tan 60^\circ = 6\sqrt{3}; \quad c = b \sec 60^\circ = 6 \times 2 = 12.$$

Also the perpendicular from  $C$  on  $AB = b \sin 60^\circ = 3\sqrt{3}$ .

$$\begin{aligned}
 11. \quad (1) \quad \text{First side} &= \cot \theta + \tan \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \operatorname{cosec} \theta \sec \theta = \operatorname{cosec} \theta \operatorname{cosec} \left( \frac{\pi}{2} - \theta \right). \\
 (2) \quad \text{First side} &= \operatorname{cosec}^2 \theta + \sec^2 \theta = \sec^2 \theta \operatorname{cosec}^2 \theta \quad [\text{Art. 31, Ex. 1.}] \\
 &= \operatorname{cosec}^2 \theta \operatorname{cosec}^2 \left( \frac{\pi}{2} - \theta \right).
 \end{aligned}$$

$$\begin{aligned}
 12. \quad &\text{In the figure on p. 41, let } PT \text{ be the pillar; then} \\
 &QR = 20 \text{ ft., } \angle PQR = 30^\circ, \angle PRT = 60^\circ, \text{ and } PR = QR = 20 \text{ ft.} \\
 &\therefore PT = PR \sin 60^\circ = \frac{20\sqrt{3}}{2} = 10\sqrt{3} = 17.32 \text{ ft.}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{First side} &= \sin^2 A \left( \frac{1}{\cos^2 A} - 1 \right) = \frac{\sin^2 A (1 - \cos^2 A)}{\cos^2 A} \\
 &= \sin^2 A \cdot \sin^2 A \cdot \sec^2 A = \sin^4 A \sec^2 A.
 \end{aligned}$$

$$14. \quad x \text{ grades} = \frac{9x}{10} \text{ degrees};$$

$$\text{also } \frac{\pi x}{300} \text{ radians} = \frac{\pi x}{300} \times \frac{180}{\pi} = \frac{3x}{5} \text{ degrees};$$

$$\therefore 3x + \frac{9x}{10} + \frac{3x}{5} = 180; \text{ whence } x = 40. \text{ Thus the angles are } 120^\circ, 36^\circ, 24^\circ.$$

$$\begin{aligned}
 15. \quad \text{Expression} &= \left( \frac{\sqrt{3}}{2} \right)^3 \sqrt{3} - 2(\sqrt{2})^2 + 3 \cdot \frac{1}{2} \cdot 1 - (\sqrt{3})^2 \\
 &= \frac{9}{8} - 4 + \frac{3}{2} - 3 = -\frac{35}{8}.
 \end{aligned}$$

$$\begin{aligned}
 16. \quad (1) \quad \text{First side} &= 1 + \tan^2 A + 2 \tan A + 1 + \cot^2 A + 2 \cot A \\
 &= \sec^2 A + \operatorname{cosec}^2 A + 2 \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \\
 &= \sec^2 A + \operatorname{cosec}^2 A + 2 \sec A \operatorname{cosec} A = (\sec A + \operatorname{cosec} A)^2. \\
 (2) \quad \text{First side} &= (\sec a - 1)^2 - \sin^2 a (\sec a - 1)^2 = (\sec a - 1)^2 (1 - \sin^2 a) \\
 &= (\sec a - 1)^2 \cos^2 a = (1 - \cos a)^2.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad (1) \quad a^2 + b^2 &> 2ab, \text{ since } (a - b)^2 \text{ is positive.} \\
 \text{Therefore } \frac{a^2 + b^2}{2ab} &> 1. \text{ Hence } \operatorname{cosec} \theta = \frac{a^2 + b^2}{2ab} \text{ is possible.}
 \end{aligned}$$

$$(2) \quad a^2 + 1 > 2a; \text{ so that } a + \frac{1}{a} > 2.$$

$$\text{Hence } 2 \sin \theta = a + \frac{1}{a} \text{ is impossible [Art. 16], unless } a = 1.$$

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18. The height of the balloon =  $660 \tan 60^\circ$  ft. =  $660\sqrt{3}$  feet.

$\therefore$  the balloon rises  $660 \times \sqrt{3}$  feet in 1.5 minutes.

$\therefore$  it rises  $\frac{660 \times \sqrt{3} \times 60}{1.5 \times 3 \times 1760}$ , or 8.66 miles per hour.

19. Let  $x^\circ$  be the common difference between the angles,  
then they are  $36^\circ, 36^\circ + x^\circ, 36^\circ + 2x^\circ$ ,

$$\therefore 3x + 3 \times 36 = 180; \text{ whence } x = 24.$$

$\therefore$  the angles are  $36^\circ, 60^\circ, 84^\circ$ , or  $\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}$  radians.

20. First side =  $\sin^2 \alpha (1 + \tan^2 \beta) + \tan^2 \beta (1 - \sin^2 \alpha) = \sin^2 \alpha + \tan^2 \beta$ .

21. Let  $CD = x =$  the perpendicular.

Then  $\angle CBD = 180^\circ - 116^\circ 33' = 63^\circ 27'$ ;  $\therefore x = DB \tan 63^\circ 27' = 2DB$ .

And  $x = DA \tan 42^\circ = \left(\frac{x}{2} + 55\right) \times .9$ ; whence  $x = 4.5 \times 20 = 90$ .

22. (1) First side =  $\frac{\cos \alpha (1 + \cos \alpha) + \sin^2 \alpha}{\sin \alpha (1 + \cos \alpha)} = \frac{\cos \alpha + 1}{\sin \alpha (1 + \cos \alpha)} = \operatorname{cosec} \alpha$ .

(2) First side =  $\frac{1 - \cos \alpha}{\sin \alpha} \left( \frac{1}{\cos \alpha} - \cos \alpha \right) = (1 - \cos \alpha) \frac{\sin \alpha}{\cos \alpha} = \tan \alpha - \sin \alpha$ .

23. First side =  $\left( \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \right)^2 = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} = \frac{1 + \cos 30^\circ}{1 - \cos 30^\circ}$ .

24. Let the man start from  $A$  and walk to  $B$ , and let  $C$  denote the position of the windmill.

Then we have  $\angle ACB = 90^\circ$ ,  $\angle CAB = 30^\circ$ ,  $BC = 1$  mile.

$\therefore AB = BC \operatorname{cosec} 30^\circ = 2$  miles.  $AC = BC \tan 60^\circ = 1.732$  miles.

And rate of walking is 2 miles per half hour, or 4 miles an hour.

25. The complement of  $\frac{3\pi}{8} = \frac{\pi}{2} - \frac{3\pi}{8} = \frac{\pi}{8}$  radians.

26. (1)  $3 \sin \theta + 4 - 4 \sin^2 \theta = \frac{9}{2}$ , (2)  $\tan \theta + \frac{2}{\sqrt{3}} = \frac{1}{\tan \theta}$ ,  
 $8 \sin^2 \theta - 6 \sin \theta + 1 = 0$ ;  $\sqrt{3} \tan^2 \theta + 2 \tan \theta - \sqrt{3} = 0$ ;  
 $\therefore (4 \sin \theta - 1)(2 \sin \theta - 1) = 0$ ;  $\therefore (\sqrt{3} \tan \theta - 1)(\tan \theta + \sqrt{3}) = 0$ ;  
 $\therefore \sin \theta = \frac{1}{4}$ , or  $\frac{1}{2}$ ,  $\therefore \tan \theta = \frac{1}{\sqrt{3}}$ , or  $-\sqrt{3}$ .  
 $\therefore \theta = 30^\circ$ , or  $14^\circ 29'$ .  $\therefore \theta = 30^\circ$ .

$$27. \text{ Here } \frac{5 \sin a - 3 \cos a}{\sin a + 2 \cos a} = \frac{5 \tan a - 3}{\tan a + 2} = \frac{20 - 15}{4 + 10} = \frac{5}{14}.$$

$$28. \text{ First side} = \frac{1 - \sin A \cos A}{\cos A \cdot \frac{\sin A - \cos A}{\sin A \cos A}} \times \frac{\sin A - \cos A}{\sin^2 A - \sin A \cos A + \cos^2 A} \\ = \frac{\sin A (1 - \sin A \cos A)}{1 - \sin A \cos A} = \sin A.$$

$$29. \text{ The distance} = 195.2 \operatorname{cosec} 77^\circ 26' \text{ yds.} = \frac{195.2}{.976} = 200 \text{ yds.}$$

$$30. \text{ We have } 70^\circ = \frac{7\pi}{18} \text{ radians.}$$

$$\therefore \text{ distance required} = 27 \times \frac{7\pi}{18} \text{ feet} = \frac{21}{2} \times \frac{22}{7} = 33 \text{ feet.}$$

### EXAMPLES. VIII. a. PAGE 70.

$$18. \sin 420^\circ = \sin (360^\circ + 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

$$20. \tan (-315^\circ) = \tan (-360^\circ + 45^\circ) = \tan 45^\circ = 1.$$

$$22. \operatorname{cosec} (-330^\circ) = \operatorname{cosec} (-360^\circ + 30^\circ) = \operatorname{cosec} 30^\circ = 2.$$

$$24. \cot \frac{17\pi}{4} = \cot \left( 4\pi + \frac{\pi}{4} \right) = \cot \frac{\pi}{4} = 1.$$

$$26. \tan \left( -\frac{5\pi}{3} \right) = \tan \left( -2\pi + \frac{\pi}{3} \right) = \tan \frac{\pi}{3} = \sqrt{3}.$$

### EXAMPLES. VIII. b. PAGE 72.

1. The boundary line of  $120^\circ$  lies in the second quadrant,

$$\therefore \cos 120^\circ = -\sqrt{1 - \sin^2 120^\circ} = -\sqrt{1 - \frac{3}{4}} = -\frac{1}{2};$$

$$\therefore \tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ} = -\sqrt{3}.$$

2. The boundary line of  $135^\circ$  lies in the second quadrant:

$$\therefore \sec 135^\circ = -\sqrt{1 + \tan^2 135^\circ} = -\sqrt{2};$$

$$\sin 135^\circ \sec 135^\circ = -1; \therefore \sin 135^\circ = \frac{1}{\sqrt{2}}.$$

and

3. The boundary line of  $240^\circ$  lies in the third quadrant;

$$\therefore \sec 240^\circ = -\sqrt{1 + \tan^2 240^\circ} = -2; \quad \therefore \cos 240^\circ = -\frac{1}{2}.$$

4. The boundary line of  $202^\circ 37'$  lies in the third quadrant;

$$\therefore \cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{12}{5}.$$

5. The boundary line of  $143^\circ 8'$  lies in the second quadrant; and

$$\operatorname{cosec} A = 1\frac{2}{3}; \quad \therefore \sin A = \frac{3}{5}; \quad \therefore \cos A = -\sqrt{1 - \sin^2 A} = -\frac{4}{5};$$

$$\therefore \sec A = -\frac{5}{4}. \quad \text{Hence } \tan A = \frac{\sin A}{\cos A} = -\frac{3}{4}.$$

6. The boundary of  $216^\circ 52'$  lies in the third quadrant;

$$\therefore \sin A = -\sqrt{1 - \cos^2 A} = -\sqrt{1 - \frac{16}{25}} = -\frac{3}{5}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{4}{3}.$$

7. The boundary line of  $\frac{2\pi}{3}$  lies in the second quadrant;

$$\text{and } \sec \frac{2\pi}{3} = -2; \quad \therefore \cos \frac{2\pi}{3} = -\frac{1}{2}. \quad \therefore \sin \frac{2\pi}{3} = +\sqrt{1 - \cos^2 \frac{2\pi}{3}} = \frac{\sqrt{3}}{2}.$$

$$\cot \frac{2\pi}{3} = \frac{\cos \frac{2\pi}{3}}{\sin \frac{2\pi}{3}} = -\frac{1}{\sqrt{3}}.$$

8. The boundary line of  $\frac{5\pi}{4}$  lies in the third quadrant;

$$\therefore \cos \frac{5\pi}{4} = -\sqrt{1 - \sin^2 \frac{5\pi}{4}} = -\frac{1}{\sqrt{2}};$$

$$\therefore \sec \frac{5\pi}{4} = -\sqrt{2}, \quad \text{and } \tan \frac{5\pi}{4} = \frac{\sin \frac{5\pi}{4}}{\cos \frac{5\pi}{4}} = 1.$$

9. We have  $\sin A = \pm \sqrt{1 - \cos^2 A} = \pm \sqrt{1 - \frac{144}{169}} = \pm \frac{5}{13}.$

$$\therefore \tan A = \frac{\sin A}{\cos A} = \pm \frac{5}{12}.$$

## EXAMPLES. IX. PAGE 79.

6. Expression  $= 1 \cdot (-1)^2 - 2(-1) \cdot 1 = 1 + 2 = 3$ .  
 7. Expression  $= 3 \cdot 0 \cdot (-1) + 2 \cdot 1 - 1 = 2 - 1 = 1$ .  
 8. Expression  $= 2 \cdot (-1)^2 \cdot 1 + 3(-1)^3 - 1 = 2 - 3 - 1 = -2$ .  
 9. Expression  $= 0 \times 0 + 1 - (-1) = 1 + 1 = 2$ .

## MISCELLANEOUS EXAMPLES. C. PAGE 80.

1. If  $\tan A = -\frac{3}{4}$ , the boundary of  $A$  will lie either in second or fourth quadrant. [See figure on page 72.]

In either position the radius vector  $= \sqrt{3^2 + 4^2} = 5$ .

Hence  $\cos NOP = -\frac{4}{5}$ ;  $\cos NOP' = \frac{4}{5}$ .

2. First side  $= (2 + \sin A)(1 - 2 \sin A) \sec A$   
 $= (2 - 3 \sin A - 2 \sin^2 A) \sec A$   
 $= (2 \cos^2 A - 3 \sin A) \sec A = 2 \cos A - 3 \tan A$ .

3.  $a = \sqrt{c^2 - b^2} = \sqrt{(21)^2 - (10 \cdot 5)^2} = 21 \sqrt{1 - \frac{1}{4}} = \frac{21\sqrt{3}}{2}$ .

$$\sin A = \frac{a}{c} = \frac{\sqrt{3}}{2}; \text{ whence } A = 60^\circ, B = 30^\circ.$$

4.  $A$  lies between  $180^\circ$  and  $270^\circ$ ;

$$\therefore \tan A = + \sqrt{\sec^2 A - 1} = \sqrt{\left(\frac{25}{7}\right)^2 - 1} = \frac{24}{7}; \text{ and } \cot A = \frac{7}{24}.$$

5. We have  $19^\circ = \frac{19\pi}{180}$  radians.

Also the radius of the earth  $= 3960$  miles.

$$\therefore \text{required distance} = \frac{19\pi}{180} \times 3960 = 19 \times 22 \times \pi = 1313 \text{ miles nearly.}$$

6. Let  $AB$  represent the cliff and  $P, Q$  the positions of the two boats.

Then  $AB = 200$  ft.,  $\angle APB = 34^\circ 30'$ ,  $\angle AQB = 18^\circ 40'$ ;

$$\therefore QB = AB \cot 18^\circ 40' = 200 \times 2.96 = 592 \text{ ft.}$$

$$PB = AB \cot 34^\circ 30' = 200 \times 1.455 = 291 \text{ ft.}$$

$$\therefore \text{required distance} = QB - PB = 301 \text{ ft.}$$

7. Since the boundary line of  $A$  is in the third quadrant,

$$\therefore \sec A = -\sqrt{1 + \tan^2 A} = -\sqrt{1 + \frac{16}{9}} = -\frac{5}{3},$$

$$\therefore \cos A = -\frac{3}{5}, \text{ and } \sin A = -\frac{4}{5}.$$

$$\therefore 2 \cot A - 5 \cos A + \sin A = 2 \cdot \frac{3}{4} - 5 \left( -\frac{3}{5} \right) - \frac{4}{5} = \frac{3}{2} + 3 - \frac{4}{5} = 3\frac{7}{10}.$$

8. We have  $71^\circ 36' 3.6'' = 71.601^\circ = \frac{71.601\pi}{180}$  radians.

$$\therefore \text{required radius} = 15 \div \frac{71.601\pi}{180} = \frac{15 \times 180}{71.601\pi} = 12.003 \text{ inches.}$$

$$9. \text{ First side} = \frac{\tan^3 \theta}{\sec^2 \theta} + \frac{\cot^3 \theta}{\operatorname{cosec}^2 \theta}$$

$$= \frac{\sin^3 \theta}{\cos \theta} + \frac{\cos^3 \theta}{\sin \theta} = \frac{\sin^4 \theta + \cos^4 \theta}{\sin \theta \cos \theta}$$

$$= \frac{(\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1 - 2 \sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta}.$$

10. Let  $AB$  represent the flagstaff,  $BC$  the tower, and let  $D$  be the position of the observer.

$$\text{Then } \angle BDC = 68^\circ 11', \quad \angle ADB = 2^\circ 10'; \quad \therefore \angle ADC = 70^\circ 21'.$$

$$\text{Let } BC = x \text{ ft., then } x + 24 = DC \tan 70^\circ 21', \text{ and } DC = x \cot 68^\circ 11'.$$

$$\therefore x + 24 = x \cot 68^\circ 11' \tan 70^\circ 21'; \quad \therefore x + 24 = x \times 2.8 \times .4 = 1.12x;$$

whence  $x = 200$ ; that is, the height of the tower is 200 ft.

11. If  $\tan A = .5$ , and  $\tan B = .3333$ , from the Tables we have  $A = 26^\circ 34'$ ,  $B = 18^\circ 26'$ ;  $\therefore A + B = 45^\circ$ .

$$12. \text{ We have } (4 \tan \theta - 3)(3 \tan \theta + 4) = 0;$$

whence

$$\tan \theta = .75, \text{ or } -1.3333.$$

$$\therefore, \text{ from the Tables, } \theta = 36^\circ 52', \text{ or } 180^\circ - 53^\circ 8',$$

that is,

$$\theta = 36^\circ 52', \text{ or } 126^\circ 52'.$$

$$13. \text{ We have } \frac{3.7}{r} = \text{radian measure of } 21^\circ 12'$$

$$= \frac{\pi}{180} \times 21.2;$$

$$\therefore 180 \times 3.7 = r \times 3.1416 \times 21.2,$$

or

$$666 = r \times 66.602, \text{ approx.}$$

$$\therefore \text{radius} = 10 \text{ in., to the nearest inch.}$$

If  $d$  be the number of miles between the two places

$$\frac{d}{4000} = \frac{3.7}{10}; \text{ whence } d = 1480.$$

14. Let  $T$  be the tower, and  $O_1, O_2$  the two points of observation. Then it is easily seen that

$$\angle TO_1O_2 = 30^\circ, \angle TO_2O_1 = 60^\circ;$$

$$\therefore \angle O_1TO_2 = 90^\circ; \text{ also } TO_2 = 2 \text{ km.}$$

$$\text{Now } O_1T = O_2T \tan 60^\circ = 2 \times 1.732 = 3.464 \text{ km.}$$

Also  $O_1O_2 = O_2T \sec 60^\circ = 4 \text{ km.}$ ; and since he walks this distance in 40 min. his rate of walking is 6 km. per hour.

$$15. \text{ From the Tables, } \alpha + \beta = 51^\circ 33', \alpha - \beta = 47^\circ 5',$$

$$\therefore \alpha = 49^\circ 19', \beta = 2^\circ 14'.$$

$$16. \text{ The expression } = \frac{10 - 6 \cot \alpha}{4 + 3 \cot \alpha} = \frac{10 - 4}{4 + 2} = 1.$$

17. Let  $F$  be the fort, and  $S_1, S_2$  the two positions of the ship. Then it is easily seen that  $\angle FS_2S_1 = 90^\circ, \angle S_2S_1F = 43^\circ$ ; also  $S_1S_2 = 20 \text{ mi.}$

$$\therefore FS_2 = 20 \tan 43^\circ = 20 \times .9325 = 18.65 \text{ mi.}$$

$$FS_1 = \frac{20}{\cos 43^\circ} = \frac{20}{.7314} = 27.346 \text{ mi.}$$

### EXAMPLES. X. a. PAGE 87.

$$1. \cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}.$$

$$2. \sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$$

$$3. \tan 240^\circ = \tan (180^\circ + 60^\circ) = \tan 60^\circ = \sqrt{3}.$$

$$4. \operatorname{cosec} 225^\circ = \operatorname{cosec} (180^\circ + 45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}.$$

$$5. \sin (-120^\circ) = -\sin 120^\circ = -\sin (180^\circ - 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}.$$

$$6. \cot (-135^\circ) = -\cot 135^\circ = -\cot (180^\circ - 45^\circ) = \cot 45^\circ = 1.$$

$$7. \cot 315^\circ = \cot (180^\circ + 135^\circ) = \cot 135^\circ = -.$$

$$8. \cos (-240^\circ) = \cos 240^\circ = \cos (180^\circ + 60^\circ) = -\cos 60^\circ = -\frac{1}{2}.$$

9.  $\sec(-300^\circ) = \sec 300^\circ = \sec(180^\circ + 120^\circ) = -\sec 120^\circ = -\sec(180^\circ - 60^\circ) = \sec 60^\circ = 2.$
10.  $\tan \frac{3\pi}{4} = \tan \left( \pi - \frac{\pi}{4} \right) = -\tan \frac{\pi}{4} = -1.$
11.  $\sin \frac{4\pi}{3} = \sin \left( \pi + \frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$
12.  $\sec \frac{2\pi}{3} = \sec \left( \pi - \frac{\pi}{3} \right) = -\sec \frac{\pi}{3} = -2.$
13.  $\operatorname{cosec} \left( -\frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6} = -2.$
14.  $\cos \left( -\frac{3\pi}{4} \right) = \cos \frac{3\pi}{4} = \cos \left( \pi - \frac{\pi}{4} \right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$
15.  $\cot \left( -\frac{5\pi}{6} \right) = -\cot \frac{5\pi}{6} = -\cot \left( \pi - \frac{\pi}{6} \right) = \cot \frac{\pi}{6} = \sqrt{3}.$
16.  $\cos(270^\circ + A) = \cos(180^\circ + 90^\circ + A) = -\cos(90^\circ + A) = \sin A.$
17.  $\cot(270^\circ - A) = \cot(180^\circ + 90^\circ - A) = \cot(90^\circ - A) = \tan A.$
18.  $\sin(A - 90^\circ) = -\sin(90^\circ - A) = -\cos A.$
19.  $\sec(A - 180^\circ) = \sec(180^\circ - A) = -\sec A.$
20.  $\sin(270^\circ - A) = \sin(180^\circ + 90^\circ - A) = -\sin(90^\circ - A) = -\cos A.$
21.  $\cot(A - 90^\circ) = -\cot(90^\circ - A) = -\tan A.$
22.  $\sin \left( \theta - \frac{\pi}{2} \right) = -\sin \left( \frac{\pi}{2} - \theta \right) = -\cos \theta.$
23.  $\tan(\theta - \pi) = -\tan(\pi - \theta) = \tan \theta.$
24.  $\sec \left( \frac{3\pi}{2} - \theta \right) = \sec \left( \pi + \frac{\pi}{2} - \theta \right) = -\sec \left( \frac{\pi}{2} - \theta \right) = -\operatorname{cosec} \theta.$
25. Expression  $= \tan A \cos A \operatorname{cosec} A = 1.$
26. Expression  $= -\sin A + \sin A - (-\sin A) - (-\sin A) = 2 \sin A.$
27. Expression  $= \sec^2 A - \tan^2 A = 1.$

### EXAMPLES. X. b. PAGE 91.

1.  $\cos 480^\circ = \cos(360^\circ + 120^\circ) = \cos 120^\circ = -\frac{1}{2}.$
2.  $\sin 960^\circ = \sin(3 \times 360^\circ - 120^\circ) = -\sin 120^\circ = -\frac{\sqrt{3}}{2}.$
3.  $\cos 780^\circ = \cos(2 \times 360^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}.$

4.  $\sin (-870^\circ) = \sin (-2 \times 360^\circ - 150^\circ) = -\sin 150^\circ = -\frac{1}{2}.$
5.  $\sec 900^\circ = \sec (2 \times 360^\circ + 180^\circ) = \sec 180^\circ = -1.$
6.  $\tan (-855^\circ) = -\tan 855^\circ = -\tan (2 \times 360^\circ + 135^\circ)$   
 $= -\tan 135^\circ = -\tan (180^\circ - 45^\circ) = \tan 45^\circ = 1$
7.  $\operatorname{cosec}(-660^\circ) = -\operatorname{cosec} 660^\circ = -\operatorname{cosec} (2 \times 360^\circ - 60^\circ) = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$
8.  $\cot 840^\circ = \cot (2 \times 360^\circ + 120^\circ) = \cot 120^\circ$   
 $= \cot (180^\circ - 60^\circ) = -\cot 60^\circ = -\frac{1}{\sqrt{3}}.$
9.  $\operatorname{cosec}(-765^\circ) = -\operatorname{cosec} 765^\circ$   
 $= -\operatorname{cosec} (2 \times 360^\circ + 45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}.$
10.  $\cos 1125^\circ = \cos (3 \times 360^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}.$
11.  $\cot 990^\circ = \cot (3 \times 360^\circ - 90^\circ) = -\cot 90^\circ = 0.$
12.  $\sin 855^\circ = \sin (2 \times 360^\circ + 135^\circ) = \sin 135^\circ = \sin (180^\circ - 45^\circ)$   
 $= \sin 45^\circ = \frac{1}{\sqrt{2}}.$
13.  $\sec 1305^\circ = \sec (4 \times 360^\circ - 135^\circ) = \sec 135^\circ = -\sec 45^\circ = -\sqrt{2}.$
14.  $\cos 960^\circ = \cos (3 \times 360^\circ - 120^\circ) = \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}.$
15.  $\sec (-1575^\circ) = \sec 1575^\circ = \sec (4 \times 360^\circ + 135^\circ)$   
 $= \sec 135^\circ = -\sec 45^\circ = -\sqrt{2}.$
16.  $\sin \frac{15\pi}{4} = \sin \left( 4\pi - \frac{\pi}{4} \right) = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$
17.  $\cot \frac{23\pi}{4} = \cot \left( 6\pi - \frac{\pi}{4} \right) = -\cot \frac{\pi}{4} = -1.$
18.  $\sec \frac{7\pi}{3} = \sec \left( 2\pi + \frac{\pi}{3} \right) = \sec \frac{\pi}{3} = 2.$
19.  $\cot \frac{16\pi}{3} = \cot \left( 6\pi - \frac{2\pi}{3} \right) = -\cot \frac{2\pi}{3} = \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}.$
20.  $\sec \left( \frac{3\pi}{2} + \frac{\pi}{3} \right) = \sec \left( 2\pi + \frac{\pi}{3} - \frac{\pi}{2} \right) = \sec \left( 2\pi - \frac{\pi}{6} \right) = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}.$
21.  $\cos \theta = \frac{\sqrt{3}}{2} = \cos 30^\circ ; \therefore \theta = 30^\circ$  satisfies the equation.

And  $\cos 30^\circ = \cos (360^\circ - 30^\circ) = \cos 330^\circ.$

There are no angles whose boundary lines are in the second and third

quadrants which satisfy the equation since the cosine in those quadrants is negative.

$\therefore$  the positive angles are  $30^\circ, 330^\circ$ .

And the negative angles are  $-(360^\circ - 30^\circ), -(360^\circ - 330^\circ)$ .

That is, the angles are  $\pm 30^\circ, \pm 330^\circ$ .

$$22. \quad \sin \theta = -\frac{1}{2} = \sin (180^\circ + 30^\circ) = \sin 210^\circ; \quad \therefore \theta = 210^\circ \text{ is a solution.}$$

Also  $\sin (360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}; \quad \therefore \theta = 330^\circ$  is another solution.

Thus  $210^\circ, 330^\circ$  are the positive angles.

The negative angles are  $-(360^\circ - 210^\circ), -(360^\circ - 330^\circ);$

$\therefore$  the required angles are  $210^\circ, 330^\circ, -150^\circ, -30^\circ$ .

$$23. \quad \tan \theta = -\sqrt{3} = \tan (180^\circ - 60^\circ) = \tan 120^\circ; \quad \therefore \theta = 120^\circ \text{ is a solution.}$$

Also  $\tan (360^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}; \quad \therefore \theta = 300^\circ$  is another solution.

Thus  $120^\circ, 300^\circ$  are the positive angles.

The negative angles are  $-(360^\circ - 120^\circ), -(360^\circ - 300^\circ);$

$\therefore$  the required angles are  $120^\circ, 300^\circ, -240^\circ, -60^\circ$ .

$$24. \quad \cot \theta = -1 = -\cot 45^\circ = \cot 135^\circ.$$

Also  $\cot 135^\circ = \cot (180^\circ + 135^\circ) = \cot 315^\circ$ .

$\therefore$  the positive angles which satisfy the equation are  $135^\circ, 315^\circ$ .

The negative angles are  $-(360^\circ - 135^\circ), -(360^\circ - 315^\circ)$ .

$\therefore$  the required angles are  $135^\circ, 315^\circ, -45^\circ, -225^\circ$ .

25. Let the radius vector  $OP$  start from the position  $OX$  and revolve in the positive direction till it reaches the position  $OP$ , such that  $\angle POX = A$ . Then let it revolve in the negative direction through an angle of  $180^\circ$ , reaching the position  $OP'$ .

Then  $POP'$  is a straight line, and  $\angle XOP' = A - 180^\circ$ .

Draw  $PM, P'M'$  perpendicular to  $XX'$ . Then the  $\Delta's$   $OPM, OP'M'$  are geometrically equal.

$$\text{Then } \sec(A - 180^\circ) = \frac{OP'}{OM'} = -\frac{OP}{OM} = -\sec A.$$

26. Proceed as in Art. 97. Let the radius vector first revolve from  $OX$  through the angle  $A$  to the position  $OP$ . Again, let it revolve from  $OX$  through  $270^\circ$  and then further through an angle  $A$  to the position  $OP'$ ; draw  $PM, P'M'$  perpendiculars to  $XX'$ . Then from the equal  $\Delta's$   $OPM, OP'M'$ , we have

$$P'M' = -OM, \quad O'M' = PM;$$

$$\therefore \tan(270^\circ + A) = \frac{P'M'}{O'M'} = -\frac{OM}{PM} = -\cot A.$$

X.]

## CERTAIN ALLIED ANGLES.

27. Let  $OP$  be determined as before, and then let the radius vector turn back in the negative direction through an angle  $90^\circ$  to the position  $OP'$ . Draw perpendiculars as before.

$$\text{Then } \cos(A - 90^\circ) = \frac{OM'}{OP'} = \frac{PM}{OP} = \sin A.$$

$$28. \text{ First side} = \tan A - \tan A - \tan A = -\tan A = \tan(360^\circ - A).$$

$$29. \text{ First side} = \frac{\sin A}{\tan A} \cdot \frac{\tan A}{-\cot A} \cdot \frac{\cos A}{-\sin A} = \sin A.$$

$$30. \text{ Expression} = \frac{-\sin A}{-\sin A} - \frac{-\cot A}{\cot A} + \frac{\cos A}{\cos A} = 1 + 1 + 1 = 3.$$

$$31. \text{ Expression} = \frac{\operatorname{cosec} A}{-\sec A} \cdot \frac{\cos A}{-\sin A} = \frac{\cos^2 A}{\sin^2 A} = \cot^2 A.$$

$$32. \text{ Expression} = \frac{-\sin A \cdot \sec A \cdot (-\tan A)}{\sec A (-\sin A) \tan A} = -1.$$

$$33. \text{ First side} = \sin\left(\pi - \frac{\pi}{2} - \theta\right) \sin\left(\frac{\pi}{2} + \pi - \theta\right) \cot\left(\pi + \frac{\pi}{2} + \theta\right) \\ = \sin\left(\frac{\pi}{2} - \theta\right) \sin\left(\frac{3\pi}{2} - \theta\right) \cot\left(\frac{\pi}{2} + \theta\right).$$

$$34. \sin \alpha = \sin \frac{11\pi}{4} = \sin\left(2\pi + \frac{3\pi}{4}\right) = \sin \frac{3\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}};$$

$$\cos \alpha = \cos \frac{3\pi}{4} = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}. \quad \text{Also } \tan \alpha = -1.$$

$$\therefore \text{ Expression} = \frac{1}{2} - \frac{1}{2} - 2 - 2 = -4.$$

## EXAMPLES. XI. a. PAGE 97.

$$1. \sin(A + 45^\circ) = \sin A \cos 45^\circ + \cos A \sin 45^\circ = \frac{1}{\sqrt{2}}(\sin A + \cos A).$$

$$2. \cos(A + 45^\circ) = \cos A \cos 45^\circ - \sin A \sin 45^\circ = \frac{1}{\sqrt{2}}(\cos A - \sin A).$$

$$3. 2 \sin(30^\circ - A) = 2(\sin 30^\circ \cos A - \cos 30^\circ \sin A) = \cos A - \sqrt{3} \sin A.$$

$$4. \cos A = \frac{4}{5}; \quad \therefore \sin A = \frac{3}{5}, \quad \cos B = \frac{3}{5}; \quad \therefore \sin B = \frac{4}{5}.$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B = 1;$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{24}{25}.$$

$$6. \sec A = \frac{17}{8}; \quad \therefore \cos A = \frac{8}{17}, \quad \sin A = \frac{15}{17}.$$

$$\operatorname{cosec} B = \frac{5}{4}; \quad \therefore \sin B = \frac{4}{5}, \quad \cos B = \frac{3}{5}.$$

$$\therefore \sec(A+B) = \frac{1}{\cos A \cos B - \sin A \sin B} = -\frac{85}{36}.$$

$$7. \sin 75^\circ = \sin(90^\circ - 15^\circ) = \cos 15^\circ = \cos(45^\circ - 30^\circ) \\ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$8. \sin 15^\circ = \cos(90^\circ - 15^\circ) = \cos 75^\circ = \cos(45^\circ + 30^\circ) \\ = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$9. \frac{\sin(\alpha+\beta)}{\cos \alpha \cos \beta} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta.$$

$$10. \frac{\sin(\alpha-\beta)}{\sin \alpha \sin \beta} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \sin \beta} = \cot \beta - \cot \alpha.$$

$$11. \frac{\cos(\alpha-\beta)}{\cos \alpha \sin \beta} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \sin \beta} = \cot \beta + \tan \alpha.$$

$$12. \cos(A+B) \cos(A-B) \\ = (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\ = \cos^2 A - \sin^2 B.$$

$$13. \sin(A+B) \sin(A-B) \\ = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\ = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ = (1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B) \\ = \cos^2 B - \cos^2 A.$$

$$14. \cos(45^\circ - A) - \sin(45^\circ + A) = \frac{1}{\sqrt{2}} \{\cos A + \sin A - \sin A - \cos A\} = 0.$$

$$15. \cos(45^\circ + A) + \sin(A - 45^\circ) = \frac{1}{\sqrt{2}} (\cos A - \sin A + \sin A - \cos A) = 0.$$

$$16. \text{First side} = \cos A \cos B + \sin A \sin B - \sin A \cos B - \cos A \sin B \\ = (\cos A - \sin A) \cos B - (\cos A - \sin A) \sin B \\ = (\cos A - \sin A) (\cos B - \sin B).$$

$$\begin{aligned} 17. \text{ First side} &= \cos A \cos B - \sin A \sin B + \sin A \cos B - \cos A \sin B \\ &= (\cos A + \sin A) \cos B - (\cos A + \sin A) \sin B \\ &= (\cos A + \sin A) (\cos B - \sin B). \end{aligned}$$

$$\begin{aligned} 18. \text{ First side} &= 2 (\sin A \cos 45^\circ + \cos A \sin 45^\circ) (\sin A \cos 45^\circ - \cos A \sin 45^\circ) \\ &= 2 \times \frac{1}{\sqrt{2}} (\sin A + \cos A) \times \frac{1}{\sqrt{2}} (\sin A - \cos A) \\ &= \sin^2 A - \cos^2 A. \end{aligned}$$

$$\begin{aligned} 19. \text{ First side} &= 2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos a - \sin a) \times \frac{1}{\sqrt{2}} (\cos a + \sin a) \\ &= \cos^2 a - \sin^2 a. \end{aligned}$$

$$\begin{aligned} 20. \text{ First side} &= 2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos a + \sin a) \times \frac{1}{\sqrt{2}} (\cos \beta - \sin \beta) \\ &= \cos a \cos \beta + \sin a \cos \beta - \cos a \sin \beta - \sin a \sin \beta \\ &= \cos (a + \beta) + \sin (a - \beta). \end{aligned}$$

21. As in Ex. 10, it is easily shown that the first term of the expression  
 $= \tan \beta - \tan \gamma$ .

Thus the first side

$$= \tan \beta - \tan \gamma + \tan \gamma - \tan a + \tan a - \tan \beta = 0.$$

### EXAMPLES. XI. b. PAGE 100.

1. We have  $\tan A = \frac{1}{2}$ ,  $\tan B = \frac{1}{3}$ ;

$$\therefore \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1.$$

3. We have  $\cot A = \frac{5}{7}$ ,  $\cot B = \frac{7}{5}$ ;

$$\therefore \cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} = 0,$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{7}{5} - \frac{5}{7}}{1 + 1} = \frac{12}{35}.$$

5.  $\tan (45^\circ + A) = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = \frac{1 + \tan A}{1 - \tan A}.$

$$6. \quad \tan(45^\circ - A) = \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} = \frac{1 - \tan A}{1 + \tan A}.$$

$$7. \quad \cot\left(\frac{\pi}{4} - \theta\right) = \frac{\cot \frac{\pi}{4} \cot \theta + 1}{\cot \theta - \cot \frac{\pi}{4}} = \frac{\cot \theta + 1}{\cot \theta - 1}.$$

$$8. \quad \cot\left(\frac{\pi}{4} + \theta\right) = \frac{\cot \frac{\pi}{4} \cot \theta - 1}{\cot \theta + \cot \frac{\pi}{4}} = \frac{\cot \theta - 1}{\cot \theta + 1}.$$

$$9. \quad \tan 15^\circ = \tan(60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = 2 - \sqrt{3}.$$

$$10. \quad \cot 15^\circ = \cot(45^\circ - 30^\circ) = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}.$$

$$\begin{aligned} 11. \quad \cos(A + B + C) &= \cos A \cos(B + C) - \sin A \sin(B + C) \\ &= \cos A \cos B \cos C - \cos A \sin B \sin C \\ &\quad - \sin A \sin B \cos C - \sin A \cos B \sin C. \\ \sin(A - B + C) &= \sin(A - B) \cos C + \cos(A - B) \sin C \\ &= \sin A \cos B \cos C - \cos A \sin B \cos C \\ &\quad + \cos A \cos B \sin C + \sin A \sin B \sin C. \end{aligned}$$

$$\begin{aligned} 12. \quad \tan(A - B - C) &= \frac{\tan(A - B) - \tan C}{1 + \tan(A - B) \tan C} \\ &= \frac{\frac{\tan A - \tan B}{1 + \tan A \tan B} - \tan C}{1 + \frac{(\tan A - \tan B) \tan C}{1 + \tan A \tan B}} \\ &= \frac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B - \tan B \tan C + \tan C \tan A}. \end{aligned}$$

$$\begin{aligned} 13. \quad \cot(A + B + C) &= \frac{\cot(A + B) \cot C - 1}{\cot C + \cot(A + B)} \\ &= \frac{\frac{(\cot A \cot B - 1) \cot C}{\cot B + \cot A} - 1}{\cot C + \frac{\cot A \cot B - 1}{\cot B + \cot A}} \\ &= \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}. \end{aligned}$$

EXAMPLES. XI. c. PAGE 101.

1. First side  $= \cos (\overline{A + B} - B) = \cos A$ .
2. First side  $= \sin (3A - A) = \sin 2A$ .
3. First side  $= \cos (2\alpha - \alpha) = \cos \alpha$ .
4. First side  $= \cos (30^\circ + A + \overline{30^\circ - A}) = \cos 60^\circ = \frac{1}{2}$ .
5. First side  $= \sin (60^\circ - A + \overline{30^\circ + A}) = \sin 90^\circ = 1$ .
6. First side  $= \cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha = \cos (2\alpha + \alpha) = \cos 3\alpha$ .
7. First side  $= \tan (\overline{\alpha - \beta} + \beta) = \tan \alpha$ .
8. First side  $= \cot (\overline{\alpha + \beta} - \alpha) = \cot \beta$ .
9. First side  $= \tan (4A - 3A) = \tan A$ .
10. 
$$\cot \theta - \cot 2\theta = \frac{\cos \theta}{\sin \theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta \cos \theta - \cos 2\theta \sin \theta}{\sin \theta \sin 2\theta}$$
$$= \frac{\sin (2\theta - \theta)}{\sin \theta \sin 2\theta} = \frac{\sin \theta}{\sin \theta \sin 2\theta} = \operatorname{cosec} 2\theta.$$
11. 
$$1 + \tan 2\theta \tan \theta = 1 + \frac{\sin 2\theta \sin \theta}{\cos 2\theta \cos \theta} = \frac{\cos \theta \sin 2\theta + \sin \theta \sin 2\theta}{\cos \theta \cos 2\theta}$$
$$= \frac{\cos (2\theta - \theta)}{\cos \theta \cos 2\theta} = \sec 2\theta.$$
12. 
$$1 + \cot 2\theta \cot \theta = \frac{\sin 2\theta \sin \theta + \cos 2\theta \cos \theta}{\sin \theta \sin 2\theta}$$
$$= \frac{\cos (2\theta - \theta)}{\sin \theta \sin 2\theta} = \operatorname{cosec} 2\theta \cot \theta.$$
13. First side  $= \sin (2\theta + \theta) = \sin 3\theta$ 
$$= \sin (4\theta - \theta)$$
$$= \sin 4\theta \cos \theta - \cos 4\theta \sin \theta.$$
14. First side  $= \cos (4\alpha + \alpha) = \cos 5\alpha$ 
$$= \cos (3\alpha + 2\alpha)$$
$$= \cos 3\alpha \cos 2\alpha - \sin 3\alpha \sin 2\alpha.$$

EXAMPLES. XI. d. PAGE 104.

1. Here  $\cos 2A = 2 \cos^2 A - 1 = -\frac{7}{9}$ .

3. We have  $\sin A = \frac{3}{5}$ , and  $\cos A = \frac{4}{5}$ ;

$$\therefore \sin 2A = 2 \sin A \cos A = \frac{24}{25}.$$

5. By Art. 124,  $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{7}{25}$ ,

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{24}{25}.$$

7. See Example, Art. 122.

8.  $\frac{\sin 2A}{1 + \cos 2A} = \frac{2 \sin A \cos A}{2 \cos^2 A} = \tan A.$

9.  $\frac{\sin 2A}{1 - \cos 2A} = \frac{2 \sin A \cos A}{2 \sin^2 A} = \cot A.$

10.  $\frac{1 - \cos A}{\sin A} = \frac{2 \sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \tan \frac{A}{2}.$

11.  $\frac{1 + \cos A}{\sin A} = \frac{2 \cos^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \cot \frac{A}{2}.$

12.  $2 \operatorname{cosec} 2a = \frac{2}{\sin 2a} = \frac{1}{\sin a \cos a} = \sec a \operatorname{cosec} a.$

13.  $\tan a + \cot a = \frac{\sin^2 a + \cos^2 a}{\sin a \cos a} = \frac{1}{\sin a \cos a} = 2 \operatorname{cosec} 2a.$

14.  $\cos^4 a - \sin^4 a = (\cos^2 a + \sin^2 a)(\cos^2 a - \sin^2 a) = \cos 2a.$

15.  $\cot a - \tan a = \frac{\cos^2 a - \sin^2 a}{\sin a \cos a} = \frac{2 \cos 2a}{2 \sin a \cos a} = 2 \cot 2a.$

16. By Art. 116,  $\cot 2A = \frac{\cot A \cot A - 1}{\cot A + \cot A} = \frac{\cot^2 A - 1}{2 \cot A}.$

17.  $\frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\frac{1}{\tan A} - \tan A}{\frac{1}{\tan A} + \tan A} = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A.$

18.  $\frac{1 + \cot^2 A}{2 \cot A} = \frac{\sin^2 A + \cos^2 A}{2 \cot A \sin^2 A} = \frac{1}{2 \sin A \cos A} = \operatorname{cosec} 2A.$

$$19. \frac{\cot^2 A + 1}{\cot^2 A - 1} = \frac{1 + \tan^2 A}{1 - \tan^2 A} = \sec 2A.$$

$$20. \frac{1 + \sec \theta}{\sec \theta} = \frac{1}{\sec \theta} + 1 = \cos \theta + 1 = 2 \cos^2 \frac{\theta}{2}.$$

$$21. \frac{\sec \theta - 1}{\sec \theta} = 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}.$$

$$22. \frac{2 - \sec^2 \theta}{\sec^2 \theta} = \frac{2}{\sec^2 \theta} - 1 = 2 \cos^2 \theta - 1 = \cos 2\theta.$$

$$23. \frac{\operatorname{cosec}^2 \theta - 2}{\operatorname{cosec}^2 \theta} = 1 - 2 \sin^2 \theta = \cos 2\theta.$$

$$24. \text{First side} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 + \sin A.$$

$$25. \text{First side} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 - \sin A.$$

$$26. \frac{\cos 2a}{1 + \sin 2a} = \frac{\cos^2 a - \sin^2 a}{(\sin a + \cos a)^2} = \frac{\cos a - \sin a}{\cos a + \sin a} = \frac{1 - \tan a}{1 + \tan a} = \tan (45^\circ - a).$$

$$27. \frac{\cos 2a}{1 - \sin 2a} = \frac{\cos^2 a - \sin^2 a}{(\cos a - \sin a)^2} = \frac{\cos a + \sin a}{\cos a - \sin a} = \cot (45^\circ - a).$$

$$28. \sin 8A = 2 \sin 4A \cos 4A = 4 \sin 2A \cos 2A \cos 4A \\ = 8 \sin A \cos A \cos 2A \cos 4A.$$

$$29. \cos 4A = 2 \cos^2 2A - 1 \\ = 2 (2 \cos^2 A - 1)^2 - 1 \\ = 8 \cos^4 A - 8 \cos^2 A + 1.$$

$$30. \text{Second side} = \cos 2 \left( 45^\circ - \frac{A}{2} \right) = \cos (90^\circ - A) = \sin A.$$

$$31. \cos^2 \left( \frac{\pi}{4} - a \right) - \sin^2 \left( \frac{\pi}{4} - a \right) = \cos 2 \left( \frac{\pi}{4} - a \right) = \cos \left( \frac{\pi}{2} - 2a \right) = \sin 2a.$$

$$32. \text{First side} = \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A} = \frac{4 \tan A}{1 - \tan^2 A} = 2 \tan 2A.$$

$$33. \text{First side} = \frac{1 + \tan A}{1 - \tan A} + \frac{1 - \tan A}{1 + \tan A} = \frac{2(1 + \tan^2 A)}{1 - \tan^2 A} \\ = 2 \sec 2A. \quad [\text{Art. 124.}]$$

**EXAMPLES. XI. e. PAGE 106.**

$$4. \quad \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{3 \sin A - 4 \sin^3 A}{\sin A} - \frac{4 \cos^3 A - 3 \cos A}{\cos A}$$

$$= 3 - 4 \sin^2 A - 4 \cos^2 A + 3$$

$$= 6 - 4 (\sin^2 A + \cos^2 A) = 2.$$

$$5. \quad \cot 3A = \cot (2A + A) = \frac{\cot 2A \cdot \cot A - 1}{\cot 2A + \cot A}$$

$$= \frac{\frac{\cot^2 A - 1}{2 \cot A} \cdot \cot A - 1}{\frac{\cot^2 A - 1}{2 \cot A} + \cot A} = \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}.$$

$$6. \quad \text{First side} = \frac{3 \cos a + 4 \cos^3 a - 3 \cos a}{3 \sin a - 3 \sin a + 4 \sin^3 a} = \cot^3 a.$$

$$7. \quad \text{First side} = \frac{3 \sin a - 3 \sin^3 a}{3 \cos a - 3 \cos^3 a} = \frac{\sin a (1 - \sin^2 a)}{\cos a (1 - \cos^2 a)} = \cot a.$$

$$8. \quad \text{First side} = \frac{3 \cos a - 3 \cos^3 a}{\cos a} + \frac{3 \sin a - 3 \sin^3 a}{\sin a}$$

$$= 3 - 3 \cos^2 a + 3 - 3 \sin^2 a$$

$$= 6 - 3 (\cos^2 a + \sin^2 a) = 3.$$

$$9. \quad \sin 18^\circ + \sin 30^\circ = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$$

$$10. \quad \cos 36^\circ - \sin 18^\circ = \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} = \frac{1}{2}.$$

$$11. \quad \cos^2 36^\circ + \sin^2 18^\circ = \left( \frac{\sqrt{5}+1}{4} \right)^2 + \left( \frac{\sqrt{5}-1}{4} \right)^2 = \frac{3+\sqrt{5}}{8} + \frac{3-\sqrt{5}}{8} = \frac{3}{4}.$$

$$12. \quad 4 \sin 18^\circ \cos 36^\circ = 4 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} = \frac{5-1}{4} = 1.$$

**EXAMPLES. XI. f. PAGE 108.**

$$1. \quad \text{First side} = \frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A} = \frac{\sin 2A \cos A - \cos 2A \sin A}{\cos A \cos 2A}$$

$$= \frac{\sin (2A - A)}{\cos A \cos 2A} = \tan A \sec 2A.$$

$$2. \quad \text{First side} = \frac{\sin 2A}{\cos 2A} + \frac{\cos A}{\sin A} = \frac{\sin 2A \sin A + \cos 2A \cos A}{\cos 2A \sin A}$$

$$= \frac{\cos A}{\cos 2A \sin A} = \cot A \sec 2A.$$

$$3. \text{ First side} = \frac{2 \sin^2 \theta + 2 \sin \theta \cos \theta}{2 \cos^2 \theta + 2 \sin \theta \cos \theta} = \frac{2 \sin \theta (\sin \theta + \cos \theta)}{2 \cos \theta (\sin \theta + \cos \theta)} = \tan \theta.$$

$$4. \text{ First side} = \frac{2 \cos^2 \frac{\theta}{2} + \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sin \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}.$$

$$\begin{aligned} 5. \quad \cos^6 a - \sin^6 a &= (\cos^2 a - \sin^2 a)(\cos^4 a + \cos^2 a \sin^2 a + \sin^4 a) \\ &= \cos 2a [(\cos^2 a + \sin^2 a)^2 - \sin^2 a \cos^2 a] \\ &= \cos 2a \left(1 - \frac{1}{4} \sin^2 2a\right). \end{aligned}$$

$$\begin{aligned} 6. \quad 4(\cos^6 \theta + \sin^6 \theta) &= 4(\cos^2 \theta + \sin^2 \theta)(\cos^4 \theta + \sin^4 \theta - \sin^2 \theta \cos^2 \theta) \\ &= 4 \{(\cos^2 \theta + \sin^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta\} \\ &= 4 - 3 \sin^2 2\theta = 1 + 3(1 - \sin^2 2\theta) \\ &= 1 + 3 \cos^2 2\theta. \end{aligned}$$

$$\begin{aligned} 7. \quad \text{First side} &= \frac{4 \cos^3 a - 3 \cos a + 3 \sin a - 4 \sin^3 a}{\cos a - \sin a} \\ &= \frac{4(\cos^3 a - \sin^3 a) - 3(\cos a - \sin a)}{\cos a - \sin a} \\ &= 4(\cos^2 a + \sin a \cos a + \sin^2 a) - 3 \\ &= 4 + 2 \sin 2a - 3 = 1 + 2 \sin 2a. \end{aligned}$$

$$\begin{aligned} 8. \quad \text{First side} &= \frac{4 \cos^3 a - 3 \cos a - 3 \sin a + 4 \sin^3 a}{\cos a + \sin a} \\ &= 4(\cos^2 a - \cos a \sin a + \sin^2 a) - 3 \\ &= 4 - 2 \sin 2a - 3 = 1 - 2 \sin 2a. \end{aligned}$$

$$9. \quad \frac{\cos a + \sin a}{\cos a - \sin a} = \frac{(\cos a + \sin a)^2}{\cos^2 a - \sin^2 a} = \frac{\sin 2a + 1}{\cos 2a} = \tan 2a + \sec 2a.$$

$$10. \quad \frac{\cot a - 1}{\cot a + 1} = \frac{\cos a - \sin a}{\cos a + \sin a} = \frac{1 - \sin 2a}{\cos 2a}. \quad [\text{See Example 2, p. 107.}]$$

$$11. \quad \frac{1 + \sin \theta}{\cos \theta} = \frac{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)^2}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}}.$$

$$12. \quad \frac{\cos \theta}{1 - \sin \theta} = \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{\cot \frac{\theta}{2} + 1}{\cot \frac{\theta}{2} - 1}.$$

$$13. \sec A - \tan A = \frac{1 - \sin A}{\cos A} = \frac{1 - \tan \frac{A}{2}}{1 + \tan \frac{A}{2}} = \tan \left( 45^\circ - \frac{A}{2} \right).$$

$$14. \tan A + \sec A = \frac{\sin A + 1}{\cos A} = \frac{1 + \tan \frac{A}{2}}{1 - \tan \frac{A}{2}} = \cot \left( 45^\circ - \frac{A}{2} \right).$$

$$15. \text{Second side} = \frac{2 \sin^2 \left( \frac{\pi}{4} + \frac{\theta}{2} \right)}{2 \cos^2 \left( \frac{\pi}{4} + \frac{\theta}{2} \right)} = \frac{1 - \cos \left( \frac{\pi}{2} + \theta \right)}{1 + \cos \left( \frac{\pi}{2} + \theta \right)} = \frac{1 + \sin \theta}{1 - \sin \theta}.$$

$$16. (2 \cos A + 1)(2 \cos A - 1) = 4 \cos^2 A - 1 = 2 \cos 2A + 1.$$

$$17. \text{First side} = \frac{2 \sin A \cos A}{2 \cos^2 A} \times \frac{\cos A}{2 \cos^2 \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}} = \tan \frac{A}{2}.$$

$$18. \text{First side} = \frac{2 \sin A \cos A}{2 \sin^2 A} \times \frac{2 \sin^2 \frac{A}{2}}{\cos A} = \frac{2 \sin^2 \frac{A}{2}}{\sin A} = \tan \frac{A}{2}.$$

$$19. \text{First side} = \cos 3a (3 \sin a - \sin 3a) + \sin 3a (3 \cos a + \cos 3a) \\ = 3 (\cos 3a \sin a + \sin 3a \cos a) = 3 \sin 4a.$$

$$20. \text{First side} = \frac{1}{4} (3 \cos a + \cos 3a) \cos 3a + \frac{1}{4} (3 \sin a - \sin 3a) \sin 3a \\ - \frac{3}{4} (\cos a \cos 3a + \sin a \sin 3a) + \frac{1}{4} (\cos^2 3a - \sin^2 3a) \\ = \frac{1}{4} \{ 3 \cos (3a - a) + \cos 6a \} = \frac{1}{4} (3 \cos 2a + \cos 6a) = \cos^3 2a.$$

$$21. \text{First side} = 3 \cos 20^\circ + \cos 60^\circ + 3 \cos 40^\circ + \cos 120^\circ \\ = 3 (\cos 20^\circ + \cos 40^\circ).$$

$$22. \text{First side} = 3 \cos 10^\circ + \cos 30^\circ + 3 \sin 20^\circ - \sin 60^\circ \\ = 3 (\cos 10^\circ + \sin 20^\circ).$$

$$23. \tan 3A = \tan (2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

Multiply up and transpose; then we obtain

$$\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$$

$$\begin{aligned}
 24. \quad \frac{\cot \theta}{\cot \theta - \cot 3\theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} &= \frac{1}{\tan \theta - \tan 3\theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} \\
 &= \frac{\tan 3\theta}{\tan 3\theta - \tan \theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} = \frac{\tan 3\theta - \tan \theta}{\tan 3\theta - \tan \theta} = 1. \\
 25. \quad \frac{1}{\tan 3\theta + \tan \theta} - \frac{1}{\cot 3\theta + \cot \theta} &= \frac{1}{\tan 3\theta + \tan \theta} - \frac{\tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} \\
 &= \frac{1 - \tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} = \frac{1}{\tan 4\theta} = \cot 4\theta.
 \end{aligned}$$

### EXAMPLES. XII. a. PAGE 112.

For Examples 1—16, see Examples on page 111.

$$\begin{aligned}
 17. \quad 2 \cos 2\beta \cos (\alpha - \beta) &= \cos (2\beta + \alpha - \beta) + \cos (2\beta - \alpha + \beta) \\
 &= \cos (\beta + \alpha) + \cos (3\beta - \alpha). \\
 18. \quad 2 \sin 3\alpha \sin (\alpha + \beta) &= \cos (3\alpha - \alpha - \beta) - \cos (3\alpha + \alpha + \beta) \\
 &= \cos (2\alpha - \beta) - \cos (4\alpha + \beta). \\
 19. \quad 2 \sin (2\theta + \phi) \cos (\theta - 2\phi) &= \sin (2\theta + \phi + \theta - 2\phi) + \sin (2\theta + \phi - \theta + 2\phi) \\
 &= \sin (3\theta - \phi) + \sin (\theta + 3\phi). \\
 20. \quad 2 \cos (3\theta + \phi) \sin (\theta - 2\phi) &= \sin (3\theta + \phi + \theta - 2\phi) - \sin (3\theta + \phi - \theta + 2\phi) \\
 &= \sin (4\theta - \phi) - \sin (2\theta + 3\phi). \\
 21. \quad \cos (60^\circ + \alpha) \sin (60^\circ - \alpha) &= \frac{1}{2} \{ \sin 120^\circ - \sin 2\alpha \} = \frac{1}{2} \left( \frac{\sqrt{3}}{2} - \sin 2\alpha \right).
 \end{aligned}$$

### EXAMPLES. XII. b. PAGE 114.

For Examples 1—12, see Examples on page 113.

$$\begin{aligned}
 13. \quad \frac{\cos \alpha - \cos 3\alpha}{\sin 3\alpha - \sin \alpha} &= \frac{2 \sin 2\alpha \sin \alpha}{2 \cos 2\alpha \sin \alpha} = \tan 2\alpha. \\
 14. \quad \frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha} &= \frac{2 \sin \frac{5\alpha}{2} \cos \frac{\alpha}{2}}{2 \sin \frac{5\alpha}{2} \sin \frac{\alpha}{2}} = \cot \frac{\alpha}{2}. \\
 15. \quad \frac{\cos 4\theta - \cos \theta}{\sin \theta - \sin 4\theta} &= \frac{-2 \sin \frac{5\theta}{2} \sin \frac{3\theta}{2}}{-2 \cos \frac{5\theta}{2} \sin \frac{3\theta}{2}} = \tan \frac{5\theta}{2}.
 \end{aligned}$$

$$16. \frac{\cos 2\theta - \cos 12\theta}{\sin 12\theta + \sin 2\theta} = \frac{2 \sin 7\theta \sin 5\theta}{2 \sin 7\theta \cos 5\theta} = \tan 5\theta.$$

$$17. \text{ First side} = 2 \cos 60^\circ \sin A = 2 \times \frac{1}{2} \sin A = \sin A.$$

$$18. \text{ First side} = 2 \cos 30^\circ \cos A = \sqrt{3} \cos A.$$

$$19. \text{ First side} = 2 \sin \frac{\pi}{4} \sin (-a) = -2 \times \frac{1}{\sqrt{2}} \sin a = -\sqrt{2} \sin a.$$

$$20. \text{ First side} = \frac{2 \cos a \cos (a - 3\beta)}{2 \sin a \cos (a - 3\beta)} = \cot a.$$

$$21. \text{ First side} = \frac{2 \sin (2\theta - \phi) \sin (\theta + 2\phi)}{2 \sin (2\theta - \phi) \cos (\theta + 2\phi)} = \tan (\theta + 2\phi).$$

$$22. \text{ First side} = \frac{2 \cos \frac{a + 5\beta}{2} \sin \frac{a - 3\beta}{2}}{2 \cos \frac{a + 5\beta}{2} \cos \frac{a - 3\beta}{2}} = \tan \frac{a - 3\beta}{2}.$$

### EXAMPLES. XII. c. PAGE 116.

$$1. \cos 3A + \sin 2A - \sin 4A = \cos 3A - 2 \cos 3A \sin A \\ = \cos 3A (1 - 2 \sin A).$$

$$2. \sin 3\theta - \sin \theta - \sin 5\theta = \sin 3\theta - 2 \sin 3\theta \cos 2\theta \\ = \sin 3\theta (1 - 2 \cos 2\theta).$$

$$3. \cos \theta + \cos 2\theta + \cos 5\theta = \cos 2\theta + 2 \cos 3\theta \cos 2\theta \\ = \cos 2\theta (1 + 2 \cos 3\theta).$$

$$4. \sin a - \sin 2a + \sin 3a = \sin a + 2 \cos \frac{5a}{2} \sin \frac{a}{2} \\ = 2 \sin \frac{a}{2} \left( \cos \frac{a}{2} + \cos \frac{5a}{2} \right) \\ = 4 \sin \frac{a}{2} \cos a \cos \frac{3a}{2}.$$

$$5. \sin 3a + \sin 7a + \sin 10a = 2 \sin 5a \cos 2a + \sin 10a \\ = 2 \sin 5a (\cos 2a + \cos 5a) \\ = 2 \sin 5a \cos \frac{7a}{2} \cos \frac{3a}{2}.$$

$$6. \sin A + 2 \sin 3A + \sin 5A = 2 \sin 3A + 2 \sin 3A \cos 2A \\ = 2 \sin 3A (1 + \cos 2A) \\ = 4 \sin 3A \cos^2 A.$$

$$7. \text{ First side} = \frac{\sin 2a + 2 \cos 3a \sin 2a}{\cos 2a + 2 \cos 3a \cos 2a} = \frac{\sin 2a (1 + 2 \cos 3a)}{\cos 2a (1 + 2 \cos 3a)} = \tan 2a.$$

$$8. \text{ First side} = \frac{(\sin a + \sin 5a) + (\sin 2a + \sin 4a)}{(\cos a + \cos 5a) + (\cos 2a + \cos 4a)} \\ = \frac{2 \sin 3a \cos 2a + 2 \sin 3a \cos a}{2 \cos 3a \cos 2a + 2 \cos 3a \cos a} = \frac{\sin 3a}{\cos 3a} = \tan 3a.$$

$$9. \text{ First side} = \frac{2 \cos 5\theta \cos 2\theta - 2 \cos 3\theta \cos 2\theta}{2 \cos 5\theta \sin 2\theta - 2 \cos 3\theta \sin 2\theta} = \cot 2\theta.$$

$$10. \text{ First side} = \frac{1}{2} (\sin 5A - \sin A) - \frac{1}{2} (\sin 5A - \sin 3A) \\ = \frac{1}{2} (\sin 3A - \sin A) = \cos 2A \sin A.$$

$$11. \text{ First side} = \frac{1}{2} (\cos 7A + \cos 3A) - \frac{1}{2} (\cos 7A + \cos A) \\ = \frac{1}{2} (\cos 3A - \cos A) = -\sin 2A \sin A.$$

$$12. \text{ First side} = \frac{1}{2} (\sin 5\theta + \sin 3\theta) - \frac{1}{2} (\sin 5\theta + \sin \theta) \\ = \frac{1}{2} (\sin 3\theta - \sin \theta) = \cos 2\theta \sin \theta.$$

$$13. \cos 5^\circ - \sin 25^\circ = \cos 5^\circ - \cos 65^\circ \\ = 2 \sin 35^\circ \sin 30^\circ = \sin 35^\circ.$$

$$14. \sin 65^\circ + \cos 65^\circ = \sin 65^\circ + \sin 25^\circ \\ = 2 \sin 45^\circ \cos 20^\circ = \sqrt{2} \cos 20^\circ.$$

$$15. \text{ First side} = 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ \\ = \cos 20^\circ - \cos 20^\circ = 0.$$

$$16. \text{ First side} = 2 \cos 48^\circ \sin 30^\circ + \cos (180^\circ - 48^\circ) \\ = \cos 48^\circ - \cos 48^\circ = 0.$$

$$17. \sin^2 5A - \sin^2 2A = \sin (5A + 2A) \sin (5A - 2A) = \sin 7A \sin 3A.$$

$$18. \cos 2A \cos 5A = \frac{1}{2} (\cos 7A + \cos 3A) = \frac{1}{2} \left( 2 \cos^2 \frac{7A}{2} - 1 + 1 - 2 \sin^2 \frac{3A}{2} \right) \\ = \cos^2 \frac{7A}{2} - \sin^2 \frac{3A}{2}.$$

$$19. \text{ First side} = 2 \sin a \cos (\beta + \gamma) + 2 \sin a \cos (\beta - \gamma) \\ = 2 \sin a \{ \cos (\beta + \gamma) + \cos (\beta - \gamma) \} \\ = 4 \sin a \cos \beta \cos \gamma.$$

$$\begin{aligned}
 20 \quad \text{First side} &= 2 \sin \gamma \sin (\alpha - \beta) + 2 \sin (\alpha + \beta) \sin \gamma \\
 &= 2 \sin \gamma \{ \sin (\alpha - \beta) + \sin (\alpha + \beta) \} \\
 &= 4 \sin \alpha \cos \beta \sin \gamma.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \text{First side} &= 2 \sin (\alpha + \beta) \cos (\alpha - \beta) - 2 \sin (\alpha + \beta) \cos (\alpha + \beta + 2\gamma) \\
 &= 2 \sin (\alpha + \beta) \{ \cos (\alpha - \beta) - \cos (\alpha + \beta + 2\gamma) \} \\
 &= 4 \sin (\alpha + \beta) \sin (\beta + \gamma) \sin (\gamma + \alpha).
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \text{First side} &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos \frac{\alpha + \beta + 2\gamma}{2} \cos \frac{\alpha + \beta}{2} \\
 &= 2 \cos \frac{\alpha + \beta}{2} \left( \cos \frac{\alpha - \beta}{2} + \cos \frac{\alpha + \beta + 2\gamma}{2} \right) \\
 &= 4 \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2} \cos \frac{\alpha + \beta}{2}.
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \text{First side} &= 2 \sin A \{ \cos 2A - \cos 120^\circ \} \\
 &= 2 \sin A \left\{ \cos 2A + \frac{1}{2} \right\} \\
 &= 2 \sin A \cos 2A + \sin A \\
 &= \sin 3A - \sin A + \sin A = \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \text{First side} &= 2 \cos \theta \left\{ \cos \frac{4\pi}{3} + \cos 2\theta \right\} = 2 \cos \theta \left( -\frac{1}{2} + \cos 2\theta \right) \\
 &= -\cos \theta + 2 \cos \theta \cos 2\theta \\
 &= -\cos \theta + \cos 3\theta + \cos \theta \\
 &= \cos 3\theta.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \text{First side} &= \cos \theta + 2 \cos \frac{2\pi}{3} \cos \theta \\
 &= \cos \theta - \cos \theta = 0.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \text{First side} &= \frac{1}{2} \{ 1 + \cos 2A + 1 + \cos 2(60^\circ + A) + 1 + \cos 2(60^\circ - A) \} \\
 &= \frac{1}{2} \{ 3 + \cos 2A + 2 \cos 120^\circ \cos 2A \} \\
 &= \frac{1}{2} \{ 3 + \cos 2A - \cos 2A \} = \frac{3}{2}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \text{First side} &= \frac{1}{2} \{ 3 - \cos 2A - \cos 2(120^\circ + A) - \cos 2(120^\circ - A) \} \\
 &= \frac{1}{2} \{ 3 - \cos 2A - 2 \cos 240^\circ \cos 2A \} \\
 &= \frac{1}{2} \{ 3 - \cos 2A + \cos 2A \} = \frac{3}{2}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \cos 20^\circ \cos 40^\circ \cos 80^\circ &= \frac{1}{2} \cos 20^\circ (\cos 120^\circ + \cos 40^\circ) \\
 &= \frac{1}{2} \cos 20^\circ \left( -\frac{3}{2} + 2 \cos^2 20^\circ \right) \\
 &= \frac{1}{4} (4 \cos^3 20^\circ - 3 \cos 20^\circ) = \frac{1}{4} \cos 60^\circ = \frac{1}{8}.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \sin 20^\circ \sin 40^\circ \sin 80^\circ &= \frac{1}{2} \sin 20^\circ \{ \cos 40^\circ - \cos 120^\circ \} \\
 &= \frac{1}{2} \sin 20^\circ \left\{ \frac{3}{2} - 2 \sin^2 20^\circ \right\} \\
 &= \frac{1}{4} \sin 60^\circ = \frac{\sqrt{3}}{8}.
 \end{aligned}$$

### EXAMPLES. XII. d. PAGE 119.

$$\begin{aligned}
 1. \quad \text{First side} &= 2 \cos (A+B) \sin (A-B) + 2 \sin C \cos C \\
 &= -2 \cos C \sin (A-B) + 2 \sin (A+B) \cos C \\
 &= 2 \cos C \{ \sin (A+B) - \sin (A-B) \} \\
 &= 4 \cos A \sin B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \text{First side} &= 2 \cos (A+B) \sin (A-B) - 2 \sin C \cos C \\
 &= -2 \cos C \{ \sin (A-B) + \sin (A+B) \} \\
 &= -4 \sin A \cos B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \text{First side} &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 &= 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right\} \\
 &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \text{First side} &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 &= 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right\} \\
 &= 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \text{First side} &= 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2} + 2 \cos^2 \frac{C}{2} - 1 \\
 &= 2 \cos \frac{C}{2} \left\{ \sin \frac{B-A}{2} + \sin \frac{B+A}{2} \right\} - 1 \\
 &= 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1.
 \end{aligned}$$

$$6. \text{ First side} = \frac{4 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

$$7. \text{ We have } \tan \frac{A+B}{2} \tan \frac{C}{2} = 1;$$

$$\therefore \frac{\left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) \tan \frac{C}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = 1.$$

Multiply up and transpose and we obtain the required result.

$$\begin{aligned} 8. \text{ First side} &= \frac{2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \cos^2 \frac{A}{2} - 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}} \\ &= \frac{\sin \frac{B+C}{2} + \sin \frac{B-C}{2}}{\sin \frac{B+C}{2} - \sin \frac{B-C}{2}} = \frac{2 \sin \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B}{2} \sin \frac{C}{2}} = \tan \frac{B}{2} \cot \frac{C}{2}. \end{aligned}$$

$$\begin{aligned} 9. \text{ First side} &= 2 \cos (A+B) \cos (A-B) + 2 \cos^2 C + 4 \cos A \cos B \cos C \\ &= -2 \cos C \{ \cos (A-B) + \cos (A+B) \} + 4 \cos A \cos B \cos C \\ &= -4 \cos A \cos B \cos C + 4 \cos A \cos B \cos C = 0. \end{aligned}$$

$$10. \text{ We have } \cot (A+B) = \cot (180^\circ - C) = -\cot C;$$

$$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C;$$

whence by multiplying up and transposing we obtain the required result.

$$\begin{aligned} 11. \text{ First side} &= \frac{\sin (B+C)}{\sin B \sin C} \cdot \frac{\sin (C+A)}{\sin C \sin A} \cdot \frac{\sin (A+B)}{\sin A \sin B} \\ &= \frac{\sin A \sin B \sin C}{\sin^2 A \sin^2 B \sin^2 C} = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C. \end{aligned}$$

$$\begin{aligned} 12. \text{ First side} &= \frac{1}{2} \{ 3 + \cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C \} \\ &= \frac{1}{2} (3 - 1), \text{ by Example 9,} \\ &= 1. \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{First side} &= \frac{1}{2} (3 - \cos A - \cos B - \cos C) \\
 &= \frac{1}{2} \left( 3 - 1 - 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \quad [\text{See Ex. 3, p. 113.}] \\
 &= 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \text{First side} &= \frac{1}{2} \{ 2 + 2 \cos^2 2A + \cos 4B + \cos 4C \} \\
 &= \frac{1}{2} \{ 2 + 2 \cos^2 2A + 2 \cos 2(B+C) \cos 2(B-C) \} \\
 &= 1 + \cos 2A \{ \cos 2(B+C) + \cos 2(B-C) \} \\
 &= 1 + 2 \cos 2A \cos 2B \cos 2C.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \text{First side} &= \frac{\cot B + \cot C}{1} + \frac{\cot C + \cot A}{1} + \frac{\cot A + \cot B}{1} \\
 &= \cot B \cot C + \cot C \cot A + \cot A \cot B \\
 &= 1. \quad [\text{See Ex. 10.}]
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \text{First side} &= \frac{\tan A \tan B \tan C}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} \\
 &= \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} \cdot \frac{1}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} \\
 &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2} \cdot 2 \sin \frac{B}{2} \cos \frac{B}{2} \cdot 2 \sin \frac{C}{2} \cos \frac{C}{2}}{\cos A \cos B \cos C \cdot 16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} \\
 &= \frac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}.
 \end{aligned}$$

### EXAMPLES. XII. e. PAGE 121.

$$1. \quad \text{First side} = \frac{1}{2} \{ \sin 2\alpha - \sin 2\beta + \sin 2\beta - \sin 2\gamma + \sin 2\gamma - \sin 2\delta + \sin 2\delta - \sin 2\alpha \} = 0.$$

$$2. \quad \text{First side} = \cot \gamma - \cot \beta + \cot \alpha - \cot \gamma + \cot \beta - \cot \alpha = 0.$$

[See Ex. XI. a, 10.]

$$\begin{aligned}
 3. \text{ First side} &= \frac{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}}{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}} \\
 &= \frac{\cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2}.
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ First side} &= \sin \alpha (\cos \beta \cos \gamma - \sin \beta \sin \gamma) - \sin \beta (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \\
 &= \cos \gamma (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \cos \gamma \sin (\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ First side} &= \cos \alpha (\cos \beta \cos \gamma - \sin \beta \sin \gamma) - \cos \beta (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \\
 &= \sin \gamma (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \sin \gamma \sin (\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 6. \text{ First side} &= \cos A \cos 2A + \sin A \sin 2A - (\sin A \cos 2A + \cos A \sin 2A) \\
 &= \cos (2A - A) - \sin (2A + A) \\
 &= \cos A - \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad a \cos 2\theta + b \sin 2\theta &= \frac{a(1 - \tan^2 \theta)}{1 + \tan^2 \theta} + \frac{2b \tan \theta}{1 + \tan^2 \theta}, & [\text{Art. 124.}] \\
 &= \frac{a(a^2 - b^2)}{a^2 + b^2} + \frac{2ab^2}{a^2 + b^2} = a \cdot \frac{a^2 + b^2}{a^2 + b^2} = a.
 \end{aligned}$$

$$8. \quad \sin 2A + \cos 2A = \frac{2 \tan A}{1 + \tan^2 A} + \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{(1 + \tan A)^2 - 2 \tan^2 A}{1 + \tan^2 A}.$$

$$9. \quad \sin 4A = 2 \sin 2A \cos 2A = \frac{4 \tan A (1 - \tan^2 A)}{(1 + \tan^2 A)^2}.$$

$$10. \quad \text{We have } A + B = 45^\circ;$$

$$\therefore \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1;$$

$$\therefore \tan A + \tan B + \tan A \tan B = 1.$$

$$\therefore (1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2.$$

$$\begin{aligned}
 11. \text{ First side} &= \frac{\cos (15^\circ - A) \cos (15^\circ + A) + \sin (15^\circ - A) \sin (15^\circ + A)}{\sin (15^\circ - A) \cos (15^\circ + A)} \\
 &= \frac{2 \cos 2A}{\sin 30^\circ - \sin 2A} = \frac{4 \cos 2A}{1 - 2 \sin 2A}.
 \end{aligned}$$

$$\begin{aligned}
 12. \text{ First side} &= \frac{\cos^2 (15^\circ + A) + \sin^2 (15^\circ + A)}{\sin (15^\circ + A) \cos (15^\circ + A)} = \frac{2}{\sin (30^\circ + 2A)} \\
 &= \frac{2}{\frac{1}{2} \cos 2A + \frac{\sqrt{3}}{2} \sin 2A} = \frac{4}{\cos 2A + \sqrt{3} \sin 2A}.
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{First side} &= \frac{\sin(A + 30^\circ) \sin(A - 30^\circ)}{\cos(A + 30^\circ) \cos(A - 30^\circ)} \\
 &= \frac{\cos 60^\circ - \cos 2A}{\cos 60^\circ + \cos 2A} = \frac{1 - 2 \cos 2A}{1 + 2 \cos 2A}.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \text{First side} &= (4 \cos^2 A - 1)(2 \cos 2A - 1) \\
 &= (2 \cos 2A + 1)(2 \cos 2A - 1) \\
 &= 4 \cos^2 2A - 1 = 2 \cos 4A + 1.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \text{We have } \tan(\theta + \phi + \psi) &= \frac{\tan \theta + \tan \phi + \tan \psi - \tan \theta \tan \phi \tan \psi}{1 - \tan \theta \tan \phi - \tan \phi \tan \psi - \tan \psi \tan \theta}, \\
 &\dots \text{ if } \theta + \phi + \psi = 0,
 \end{aligned}$$

we have  $\tan \theta + \tan \phi + \tan \psi = \tan \theta \tan \phi \tan \psi.$

Now putting  $\theta = \beta - \gamma, \phi = \gamma - \alpha, \psi = \alpha - \beta,$

we have at once the required identity.

16. First side

$$\begin{aligned}
 &= 2 \sin \frac{\beta - \alpha}{2} \cos \frac{\alpha + \beta - 2\gamma}{2} + \sin(\alpha - \beta) + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} \\
 &= -2 \sin \frac{\alpha - \beta}{2} \left\{ \cos \frac{\alpha + \beta - 2\gamma}{2} - \cos \frac{\alpha - \beta}{2} \right\} + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} \\
 &= -4 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha - \gamma}{2} \sin \frac{\gamma - \beta}{2} + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \text{First side} &= \frac{1}{2} \{ \cos 2(\beta - \gamma) + \cos 2(\gamma - \alpha) + 2 \cos^2(\alpha - \beta) + 2 \} \\
 &= \frac{1}{2} \{ 2 \cos(\alpha - \beta) \cos(\beta - 2\gamma + \alpha) + 2 \cos^2(\alpha - \beta) + 2 \} \\
 &= \cos(\alpha - \beta) [ \cos(\beta - 2\gamma + \alpha) + \cos(\alpha - \beta) ] + 1 \\
 &= 1 + 2 \cos(\beta - \gamma) \cos(\gamma - \alpha) \cos(\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \text{First side} &= 1 + \frac{1}{2} (\cos 2\alpha + \cos 2\beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\
 &= 1 + \cos(\alpha + \beta) \cos(\alpha - \beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\
 &= 1 + \cos(\alpha + \beta) \{ \sin \alpha \sin \beta - \cos \alpha \cos \beta \} \\
 &= 1 - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta).
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \text{First side} &= 1 - \frac{1}{2} (\cos 2\alpha + \cos 2\beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta) \\
 &= 1 - \cos(\alpha + \beta) \{ \cos(\alpha - \beta) - 2 \sin \alpha \sin \beta \} \\
 &= 1 - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta).
 \end{aligned}$$

20. First side  $= \cos 12^\circ + 2 \cos 72^\circ \cos 12^\circ$

$$= \cos 12^\circ \left( 1 + \frac{\sqrt{5}-1}{2} \right) = \cos 12^\circ \left( \frac{\sqrt{5}+1}{2} \right)$$

$$= 2 \cos 12^\circ \cos 36^\circ = \cos 24^\circ + \cos 48^\circ.$$

21.  $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi-A}{4} \cos \frac{\pi-B}{4} \cos \frac{\pi-C}{4}$  [Ex. 2, p. 121.]

$$= 4 \cos \frac{B+C}{4} \cos \frac{C+A}{4} \cos \frac{A+B}{4}.$$

22. Second side  $= 2 \cos \frac{\pi-B}{4} \left[ \cos \frac{2\pi+A+C}{4} + \cos \frac{A-C}{4} \right]$

$$= 2 \cos \frac{A+C}{4} \cos \left( \frac{\pi}{2} + \frac{A+C}{4} \right) + 2 \cos \frac{A+C}{4} \cos \frac{A-C}{4}$$

$$= -2 \cos \frac{A+C}{4} \sin \frac{A+C}{4} + \cos \frac{A}{2} + \cos \frac{C}{2}$$

$$= -\sin \frac{A+C}{2} + \cos \frac{A}{2} + \cos \frac{C}{2}$$

$$= \cos \frac{A}{2} - \cos \frac{B}{2} + \cos \frac{C}{2}.$$

23. This may be done in the same way as the two preceding examples. The following solution exhibits another method.

From Example 3 of Art. 135,

if  $\alpha + \beta + \gamma = \pi$ , then  $\cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$

Put  $\alpha = \frac{\pi}{2} - \frac{A}{2}, \quad \beta = \frac{\pi}{2} - \frac{B}{2}, \quad \gamma = \frac{\pi}{2} - \frac{C}{2};$

then  $\alpha + \beta + \gamma = \pi$ , and after substituting for  $\alpha, \beta, \gamma$ ,

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 1 + 4 \sin \frac{\pi-A}{4} \sin \frac{\pi-B}{4} \sin \frac{\pi-C}{4}.$$

24. First side  $= \frac{2 \sin (\alpha + \beta) \cos (\alpha - \beta) + \sin 2\gamma}{2 \sin (\alpha + \beta) \cos (\alpha - \beta) + \sin 2\gamma} = \frac{\cos (\alpha - \beta) + \sin \gamma}{\cos (\alpha - \beta) + \sin \gamma}$

$$= \frac{\cos (\alpha - \beta) + \cos (\alpha + \beta)}{\cos (\alpha - \beta) - \cos (\alpha + \beta)} = \cot \alpha \cot \beta.$$

25. Here  $\tan (\alpha + \beta) = \tan \left( \frac{\pi}{2} - \gamma \right) = \cot \gamma;$

$$\therefore \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{1}{\tan \gamma}.$$

Multiply up, and transpose.

## EXAMPLES. XII. f. PAGE 122 A.

9. Use the formulæ of Arts. 123, 124.

$$10. \quad \frac{\sin 3A}{\sin 2A - \sin A} = \frac{3 \sin A - 4 \sin^3 A}{2 \sin A \cos A - \sin A} \\ = \frac{3 - 4 \sin^2 A}{2 \cos A - 1} = \frac{4 \cos^2 A - 1}{2 \cos A - 1} = 2 \cos A + 1.$$

$$11. \quad \text{Here } \cos \alpha = \sqrt{1 - \frac{7^2}{25^2}} = \frac{24}{25}, \quad \sin \beta = \sqrt{1 - \frac{3^2}{5^2}} = \frac{4}{5}. \\ \therefore \cos(\alpha + \beta) = \frac{24}{25} \cdot \frac{6}{10} - \frac{7}{25} \cdot \frac{8}{10} = \frac{88}{250} = .352;$$

whence, by the Tables,  $\alpha + \beta = 69^\circ 23'$ .

$$\text{Again, } \sin \alpha = .28 \text{ gives } \alpha = 16^\circ 15', \\ \cos \beta = .6 \text{ gives } \beta = 53^\circ 8'; \\ \therefore \alpha + \beta = 69^\circ 23'.$$

$$12. \quad \sin(\alpha + \beta + \gamma) = \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \sin \alpha \cos(\beta + \gamma), \\ \sin(\alpha + \beta - \gamma) + \sin(\alpha - \beta + \gamma) = 2 \sin \alpha \cos(\beta - \gamma); \\ \therefore, \text{ by addition, first side} = 2 \sin \alpha \{ \cos(\beta + \gamma) + \cos(\beta - \gamma) \} \\ = 4 \sin \alpha \cos \beta \cos \gamma.$$

$$13. \quad \cos 57^\circ + \sin 27^\circ = \cos 57^\circ + \cos 63^\circ \\ = 2 \cos 60^\circ \cos 3^\circ \\ = \cos 3^\circ.$$

$$\text{Again, } \cos 57^\circ = .5446, \\ \cos 63^\circ = .4540; \\ \therefore \cos 57^\circ + \cos 63^\circ = .9986 = \cos 3^\circ, \text{ by the Tables.}$$

$$14. \quad \text{Expression} = 2 \sin 5a (\cos 5a + \cos a) \\ = \sin 10a + \sin 6a + \sin 4a.$$

$$15. \quad \text{Expression} = 2 \cos 20^\circ (\cos 70^\circ + \cos 10^\circ) \\ = \cos 90^\circ + \cos 50^\circ + \cos 30^\circ + \cos 10^\circ \\ = \cos 50^\circ + \cos 30^\circ + \cos 10^\circ \\ = .6428 + .8660 + .9848 \\ = 2.4936.$$

16. See MISCELLANEOUS EXAMPLES K. No. 242.

$$\begin{aligned}
 17. \quad (i) \quad \text{First side} &= x^2 + y^2 + xy (\cot^2 \alpha + \tan^2 \alpha) \\
 &= (x+y)^2 - 2xy + xy \left( \frac{1}{\tan^2 \alpha} + \tan^2 \alpha \right) \\
 &= (x+y)^2 + xy \left( \frac{1 + \tan^4 \alpha}{\tan^2 \alpha} - 2 \right) \\
 &= (x+y)^2 + 4xy \frac{(1 - \tan^2 \alpha)^2}{4 \tan^2 \alpha} \\
 &= (x+y)^2 + 4xy \cot^2 2\alpha.
 \end{aligned}$$

(ii) Use the formula of Ex. 12 on p. 97.

$$\begin{aligned}
 (iii) \quad &(\cos \alpha + \cos 7\alpha) + \cos (3\alpha + \cos 5\alpha) \\
 &= 2 \cos 4\alpha \cos 3\alpha + 2 \cos 4\alpha \cos \alpha \\
 &= 2 \cos 4\alpha (\cos 3\alpha + \cos \alpha) \\
 &= \frac{\sin 8\alpha}{\sin 4\alpha} \cdot 2 \cos 2\alpha \cdot \cos \alpha \\
 &= \frac{\sin 8\alpha \cdot 2 \cos 2\alpha \cos \alpha}{2 \sin 2\alpha \cos 2\alpha} \\
 &= \frac{1}{2} \sin 8\alpha \cdot \frac{2 \cos \alpha}{\sin 2\alpha} = \frac{1}{2} \sin 8\alpha \operatorname{cosec} \alpha.
 \end{aligned}$$

$$18. \quad \sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{6}{10} \cdot \frac{8}{10} = .96,$$

$$\cos 2\theta = 2 \cos^2 \theta - 1 = 2 \cdot \frac{64}{100} - 1 = .28.$$

Again,  $\cos \theta = .8$  gives  $\theta = 36^\circ 52'$ .  $\therefore 2\theta = 73^\circ 44'$ ,  
and from the Tables,  $\sin 73^\circ 44' = .96$ ,  $\cos 73^\circ 44' = .28$ .

$$\begin{aligned}
 19. \quad \text{Here} \quad \cos A &= \cos \left( 45^\circ - \frac{B}{2} \right) = \frac{1}{\sqrt{2}} \left( \cos \frac{B}{2} + \sin \frac{B}{2} \right) \\
 &= \frac{1}{\sqrt{2}} \sqrt{\cos^2 \frac{B}{2} + \sin^2 \frac{B}{2} + \sin B} = \sqrt{\frac{1 + \sin B}{2}}.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad (i) \quad \text{First side} &= \frac{\cos 10^\circ - \sqrt{3} \sin 10^\circ}{\sin 10^\circ \cos 10^\circ} = \frac{4 \left( \frac{1}{2} \cos 10^\circ - \frac{\sqrt{3}}{2} \sin 10^\circ \right)}{\sin 20^\circ} \\
 &= \frac{4 \sin (30^\circ - 10^\circ)}{\sin 20^\circ} = 4.
 \end{aligned}$$

$$(ii) \quad \text{Second side} = \sin 18^\circ + \sin 30^\circ = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$$

21. Here  $\sin C = \sin B$ ,  $\cos C = -\cos B$ .

$\therefore$  Second side  $= 2 \cos^2 B = 2(1 - \sin^2 B) = 2(1 - \sin B \sin C)$ .

22. 
$$\begin{aligned}\cos^2 B + \cos^2 C &= \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} \\ &= 1 + \frac{1}{2}(\cos 2B + \cos 2C) \\ &= 1 + \cos(B+C) \cos(B-C) \\ &= 1 - \cos A \cos(B-C), \text{ for } \cos(B+C) = -\cos A, \\ &= 1 + \cos^2 A - \cos A \{\cos A + \cos(B-C)\} \\ &= 1 + \cos^2 A - \cos A \{\cos(B-C) - \cos(B+C)\} \\ &= 1 + \cos^2 A - 2 \sin B \sin C \cos A.\end{aligned}$$

23. As in Ex. 22,

$$\begin{aligned}\cos^2 B + \cos^2 C &= 1 + \cos(B+C) \cos(B-C) \\ &= 1 + \cos A \cos(B-C), \text{ for } \cos(B+C) = \cos A, \\ &= 1 + \cos^2 A + \cos A \{\cos(B-C) - \cos(B+C)\} \\ &= 1 + \cos^2 A + 2 \sin B \sin C \cos A.\end{aligned}$$

24. 
$$\begin{aligned}\cos^2 A \cos 2B - \cos^2 B \cos 2A &= \cos^2 A (\cos^2 B - \sin^2 B) \\ &\quad - \cos^2 B (\cos^2 A - \sin^2 A) \\ &= \cos^2 B \sin^2 A - \cos^2 A \sin^2 B \\ &= (1 - \sin^2 B) \sin^2 A - (1 - \sin^2 A) \sin^2 B \\ &= \sin^2 A - \sin^2 B;\end{aligned}$$

whence, by transposition, we have the required result.

25. 
$$\begin{aligned}\tan 50^\circ - \tan 40^\circ &= \frac{\sin 50^\circ}{\cos 50^\circ} - \frac{\sin 40^\circ}{\cos 40^\circ} \\ &= \frac{\sin 50^\circ \cos 40^\circ - \cos 50^\circ \sin 40^\circ}{\cos 50^\circ \cos 40^\circ} \\ &= \frac{\sin(50^\circ - 40^\circ)}{\cos 50^\circ \cos 40^\circ} = \frac{\sin 10^\circ}{\cos 50^\circ \sin 50^\circ} \\ &= \frac{2 \sin 10^\circ}{\sin 100^\circ} = \frac{2 \sin 10^\circ}{\cos 10^\circ} = 2 \tan 10^\circ.\end{aligned}$$

26. Here  $\tan \theta = 2$ ;  $\therefore \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{4}{1+4} = \frac{4}{5} = .8$

and  $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1-4}{1+4} = -\frac{3}{5} = -.6.$

Again, from the Tables,  $\cot \theta = .5$  gives  $\theta = 63^\circ 26'$ .

$$\therefore 2\theta = 126^\circ 52'.$$

$$\therefore \sin 2\theta = \sin (180^\circ - 126^\circ 52') = \sin 53^\circ 8' \\ = .8, \text{ from the Tables.}$$

$$\cos 2\theta = -\cos 53^\circ 8' = -.6, \text{ from the Tables.}$$

27. If  $\tan \alpha = .362$ , the equation may be written

$$\tan \alpha \cos \theta + \sin \theta = 1, \text{ or } \sin \alpha \cos \theta + \cos \alpha \sin \theta = \cos \alpha.$$

$$\therefore \sin (\alpha + \theta) = \cos \alpha = \sin (90^\circ - \alpha).$$

Now from the Tables,  $\alpha = 19^\circ 54'$ .

$$\therefore \sin (19^\circ 54' + \theta) = \sin (90^\circ - 19^\circ 54') = \sin 70^\circ 6'.$$

$$\therefore 19^\circ 54' + \theta = 70^\circ 6', \text{ or } 180^\circ - 70^\circ 6';$$

whence

$$\theta = 50^\circ 12', \text{ or } 90^\circ.$$

### EXAMPLES. XIII a. PAGE 128.

$$1. \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{225 + 49 - 169}{210} = \frac{105}{210} = \frac{1}{2}; \therefore C = 60^\circ.$$

$$2. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 25 - 49}{30} = -\frac{15}{30} = -\frac{1}{2}; \therefore A = 120^\circ.$$

$$3. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2(3+1-1)}{5^2 2 \sqrt{3}} = \frac{\sqrt{3}}{2}; \therefore A = 30^\circ.$$

Also  $a = c$ ;  $\therefore C = A = 30^\circ$ ; hence  $B = 180^\circ - A - C = 120^\circ.$

$$4. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{961 + 98 - 625}{434 \sqrt{2}} = \frac{1}{\sqrt{2}}; \therefore A = 45^\circ.$$

5. Let  $a=2$ ,  $b=2\frac{2}{3}$ ,  $c=3\frac{1}{3}$ .

Then 
$$\cos C = \frac{4 + \frac{64}{9} - \frac{100}{9}}{\frac{32}{3}} = 0; \therefore C = 90^\circ.$$

$$6. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 6 - (4 + 2\sqrt{3})}{4\sqrt{6}} = \frac{3 - \sqrt{3}}{2\sqrt{6}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \therefore A = 75^\circ;$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{6 + 4 + 2\sqrt{3} - 4}{2\sqrt{6}(\sqrt{3} + 1)} = \frac{3 + \sqrt{3}}{\sqrt{2}(3 + \sqrt{3})} = \frac{1}{\sqrt{2}}; \therefore B = 45^\circ;$$

$$\therefore C = 180^\circ - 75^\circ - 45^\circ = 60^\circ.$$

$$7. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 2\sqrt{3} - 2}{4(\sqrt{3} - 1)} = \frac{3 - \sqrt{3}}{2(\sqrt{3} - 1)} = \frac{\sqrt{3}}{2}; \therefore A = 30^\circ;$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{4 - 2\sqrt{3} + 2 - 4}{2\sqrt{2}(\sqrt{3} - 1)} = \frac{1 - \sqrt{3}}{\sqrt{2}(\sqrt{3} - 1)} = -\frac{1}{\sqrt{2}};$$

$$\therefore B = 135^\circ;$$

$$\therefore C = 180^\circ - 30^\circ - 135^\circ = 15^\circ.$$

$$8. \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 25 - 19}{80} = \frac{70}{80} = .875; \therefore C = 28^\circ 57'.$$

$$9. \quad \cos C = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5} = \frac{16 + 25 - 49}{40} = -\frac{8}{40} = -\frac{1}{5} = -\cos 78^\circ 28';$$

$$\therefore C = 180^\circ - 78^\circ 28' = 101^\circ 32'.$$

$$10. \quad c^2 = a^2 + b^2 - 2ab \cos C = 4 + 4 + 2\sqrt{3} - 2 \cdot 2(\sqrt{3} + 1)\frac{1}{2} = 6.$$

$$11. \quad b^2 = c^2 + a^2 - 2ca \cos B = 9 + 25 - 2 \cdot 3 \cdot 5 \left(-\frac{1}{2}\right) = 9 + 25 + 15 = 49;$$

$$\therefore b = 7.$$

$$12. \quad a^2 = b^2 + c^2 - 2bc \cos A = 49 + 36 - 2 \cdot 7 \cdot 6 \times .2501 \\ = 49 + 36 - 21.0042 = 64, \text{ approx.}; \text{ whence } a = 8.$$

$$13. \quad a^2 = b^2 + c^2 - 2bc \cos A = 64 + 121 - 2 \cdot 8 \cdot 11 \left(-\frac{1}{16}\right) \\ = 64 + 121 + 11 = 196;$$

$$\therefore a = 14.$$

$$14. \quad b^2 = c^2 + a^2 - 2ca \cos B = 9 + 49 - 2 \cdot 3 \cdot 7 (-\cdot 5476) = 81, \text{ approx.};$$

$$\therefore b = 9.$$

$$15. \quad b^2 = c^2 + a^2 - 2ca \cos B = 48 - 24\sqrt{3} + 24 - 2 \cdot 2\sqrt{6} \cdot 2\sqrt{3} (\sqrt{3} - 1) \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} \\ = 48 - 24\sqrt{3} + 24 - 4 \cdot 3 \cdot (4 - 2\sqrt{3}) = 24;$$

$$\therefore b = 2\sqrt{6}; \text{ whence } A = B = 75^\circ, \text{ and } C = 30^\circ.$$

$$16. \quad a^2 = b^2 + c^2 - 2bc \cos A = 4 + 6 + 2\sqrt{5} - 2 \cdot 2(\sqrt{5} + 1) \frac{\sqrt{5} - 1}{4} \\ = 4 + 6 + 2\sqrt{5} - 4 = 6 + 2\sqrt{5};$$

$$\therefore a = \sqrt{5} + 1; \text{ whence } C = A = 72^\circ, \text{ and } B = 36^\circ.$$

$$17. \quad C = 75^\circ; \text{ whence } a = c.$$

Also 
$$a = \frac{b \sin A}{\sin B} = \frac{\sqrt{8}(\sqrt{3} + 1)2}{2\sqrt{2}} = 2\sqrt{3} + 2.$$

$$18. \quad c = \frac{b \sin C}{\sin B} = \sqrt{6} \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{2}{\sqrt{3}} = \sqrt{3} - 1.$$

Also  $A = 105^\circ; \text{ whence } a = \frac{b \sin A}{\sin B} = \sqrt{6} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \frac{2}{\sqrt{3}} = \sqrt{3} + 1.$

$$19. \quad C = 30^\circ; \text{ whence } a = \frac{c \sin A}{\sin C} = \frac{\sqrt{2} \times 2}{\sqrt{2}} = 2;$$

$$b = \frac{a \sin B}{\sin A} = 2 \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \sqrt{2} = \sqrt{3} + 1.$$

$$20. \quad \text{Here } \frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin 75^\circ}{\sin 45^\circ} = \frac{\sqrt{3} + 1}{2\sqrt{2}} / \frac{1}{\sqrt{2}} = \frac{\sqrt{3} + 1}{2}.$$

$$21. \quad \sin A = \frac{a \sin C}{c} = \frac{2}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2};$$

$$\therefore b = a \cos C + c \cos A = 2 \left( -\frac{1}{2} \right) + 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 3 - 1 = 2.$$

$$22. \quad \sin A = \frac{a \sin B}{b} = \frac{3}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2};$$

$$\therefore c = a \cos B + b \cos A = 3 \cdot \frac{1}{2} + 3\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 6.$$

$$23. \quad \text{We have } (b+c)^2 - a^2 = 3bc;$$

$$\therefore \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}; \text{ whence } A = 60^\circ.$$

$$24. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{12 + 6 - (12 + 6\sqrt{3})}{2 \cdot 2\sqrt{3} \cdot \sqrt{6}} \\ = \frac{6(1 - \sqrt{3})}{12\sqrt{2}} = -\frac{\sqrt{3} - 1}{2\sqrt{2}} = -\cos 75^\circ;$$

$$\therefore A = 105^\circ; \text{ similarly } C = 30^\circ.$$

$$\therefore \sin B = \frac{b \sin C}{c} = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}}; \text{ whence } B = 45^\circ.$$

$$25. \quad \text{The sides are proportional to } \sqrt{3} + 1, \sqrt{3} - 1, \sqrt{6};$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 - 2\sqrt{3} + 6 - (4 + 2\sqrt{3})}{2\sqrt{6}(\sqrt{3} - 1)} \\ = \frac{6 - 4\sqrt{3}}{2\sqrt{6}(\sqrt{3} - 1)} = \frac{2\sqrt{3}(\sqrt{3} - 2)}{2\sqrt{3} \cdot \sqrt{2}(\sqrt{3} - 1)} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \\ = \frac{1 - \sqrt{3}}{2\sqrt{2}} = -\cos 75^\circ;$$

$$\therefore A = 105^\circ.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6}}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ, \text{ and } B = 15^\circ.$$

$$26. \quad \text{Here } b = \frac{\sqrt{6} + \sqrt{2}}{4}, \quad c = \frac{\sqrt{6} - \sqrt{2}}{4}, \quad A = 60^\circ;$$

$$\therefore a^2 = \frac{(\sqrt{6} + \sqrt{2})^2 + (\sqrt{6} - \sqrt{2})^2}{16} - \frac{2 \cdot 4}{16} \cdot \frac{1}{2} = \frac{12}{16} = \frac{3}{4};$$

$$\therefore a = \frac{\sqrt{3}}{2}.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6} - \sqrt{2}}{4} \cdot \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}};$$

$$\text{whence } C = 15^\circ, \text{ and } B = 105^\circ.$$

## EXAMPLES. XIII. b. PAGE 132.

$$1. \sin B = \frac{b \sin A}{a} = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}.$$

$\therefore B = 60^\circ$  or  $120^\circ$ ; and since  $a < b$ , both these values are admissible. Hence  $C = 90^\circ$  or  $30^\circ$ .

$$c = \frac{a \sin C}{\sin A} = 2 \text{ or } 1, \text{ on reduction.}$$

$$2. \sin B = \frac{b \sin C}{c} = \frac{3\sqrt{2}}{2\sqrt{3}} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

$\therefore B = 60^\circ$  or  $120^\circ$ ; and since  $c < b$ , both these values are admissible. Hence  $A = 75^\circ$  or  $15^\circ$ .

$$a = \frac{b \sin A}{\sin B} = 3 + \sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

$$3. \sin A = \frac{a \sin C}{c} = \frac{2}{\sqrt{6}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}.$$

$\therefore A = 45^\circ$ , the other value being inadmissible, since  $c > a$ . Hence  $B = 75^\circ$ .

$$b = \frac{a \sin B}{\sin A} = \sqrt{3} + 1.$$

4.  $\sin C = \frac{c}{a} \sin A = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}$ , which is impossible. Thus there is no triangle with the given parts.

$$5. \sin C = \frac{c}{b} \sin B = \frac{2\sqrt{3}}{\sqrt{6}} \cdot \frac{1}{2} = \frac{1}{\sqrt{2}}.$$

$\therefore C = 45^\circ$  or  $135^\circ$ ; and since  $b < c$ , both values are admissible. Hence  $A = 105^\circ$  or  $15^\circ$ .

$$a = \frac{b \sin A}{\sin B} = \sqrt{3}(\sqrt{3} + 1), \text{ or } \sqrt{3}(\sqrt{3} - 1), \text{ on reduction.}$$

$$6. \sin C = \frac{c}{b} \sin B = \frac{3 + \sqrt{3}}{3\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$\therefore C = 75^\circ$  or  $105^\circ$ ; and since  $b < c$ , both values are admissible. Hence  $A = 45^\circ$  or  $15^\circ$ .

$$a = \frac{b \sin A}{\sin B} = 2\sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

$$7. \sin A = \frac{a}{c} \sin C = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$\therefore A = 75^\circ$  or  $105^\circ$ ; and since  $c < a$ , both values are admissible. Hence  $B = 90^\circ$  or  $60^\circ$ .

$$b = \frac{c \sin B}{\sin C} = 2\sqrt{6}, \text{ or } 3\sqrt{2}, \text{ on reduction.}$$

$$8. \sin B = \frac{b}{a} \sin A = \frac{4(\sqrt{5}+1)}{4} \cdot \frac{\sqrt{5}-1}{4} = 1.$$

$\therefore B = 90^\circ$ , and there is no ambiguity.

$$c^2 = b^2 - a^2 = (b+a)(b-a) = (8+\sqrt{80})\sqrt{80}$$

$$= 4(2+\sqrt{5}) \cdot 4\sqrt{5} = 16\sqrt{5}(2+\sqrt{5})$$

$$\therefore c = 4\sqrt{5+2\sqrt{5}}.$$

$$9. \sin A = \frac{a}{b} \sin B = \frac{3\sqrt{2}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

$\therefore A = 60^\circ$  or  $120^\circ$ ; but *neither* of these values is admissible as in each case the sum of the angles would be greater than  $180^\circ$ . Thus the triangle is impossible.

### EXAMPLES. XIII. c. PAGE 134.

1. Follows at once from Art. 137.

2. The first side  $= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2 = a^2 + b^2 + c^2$ .

3. The first side  $= \frac{a^2 + b^2 - c^2}{2} - \frac{a^2 + c^2 - b^2}{2} = b^2 - c^2$ .

4. The first side  $= (b \cos A + a \cos B) + (c \cos B + b \cos C)$   
 $+ (c \cos A + a \cos C) = c + a + b$ .

5. The first side  $= a(1 - \cos C) + c(1 - \cos A)$   
 $= a + c - (a \cos C + c \cos A)$   
 $= a + c - b$ .

6. The second side  $= \frac{a \cos B}{a \cos C} = \frac{\cos B}{\cos C}$ .

7. The second side  $= \frac{a \sin C}{c \cos A} = \frac{c \sin A}{c \cos A} = \tan A$ .

8. Put  $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ ; then

$$\begin{aligned} \text{the first side} &= k(\sin B + \sin C) \sin \frac{A}{2} \\ &= 2k \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2} \\ &= 2k \sin \frac{A}{2} \cos \frac{A}{2} \cdot \cos \frac{B-C}{2} \\ &= k \sin A \cdot \cos \frac{B-C}{2} \\ &= a \cos \frac{B-C}{2}. \end{aligned}$$

$$\begin{aligned}
 9. \text{ The first side} &= \frac{k(\sin A + \sin B)}{k \sin C} \sin^2 \frac{C}{2} \\
 &= \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} \sin^2 \frac{C}{2} \\
 &= \cos \frac{A-B}{2} \sin \frac{C}{2} \\
 &= \cos \frac{A-B}{2} \cos \frac{A+B}{2} \\
 &= \frac{\cos A + \cos B}{2}.
 \end{aligned}$$

$$\begin{aligned}
 10. \text{ The first side} &= k \{ \sin A \sin (B-C) + \dots + \dots \} \\
 &= k \{ \sin (B+C) \sin (B-C) + \dots + \dots \} \\
 &= k \{ \sin^2 B - \sin^2 C + \dots + \dots \} = 0.
 \end{aligned}$$

$$\begin{aligned}
 11. \text{ The second side} &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C} \\
 &= \frac{\sin (A+B) \sin (A-B)}{\sin (A+B) \sin (A+B)} = \frac{\sin (A-B)}{\sin (A+B)}.
 \end{aligned}$$

$$\begin{aligned}
 12. \text{ The second side} &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C - \sin^2 A} = \frac{\sin (A+B) \sin (A-B)}{\sin (C+A) \sin (C-A)} \\
 &= \frac{\sin C \sin (A-B)}{\sin C \sin (C-A)} = \frac{c \sin (A-B)}{b \sin (C-A)}.
 \end{aligned}$$

### EXAMPLES. XIII. d. PAGE 136.

1. Let  $ABC$  be the triangle, in which  $B=C=2A$ , and  $a=2$ ;  
then  $B+C+A=5A=180^\circ$ ;  $\therefore A=36^\circ$ ,  $B=C=72^\circ$ ;

$$\text{and } b=c = \frac{a}{\sin A} \sin B = \frac{2 \sin 2A}{\sin A} = 2a \cos A = 4 \cdot \frac{\sqrt{5}+1}{4} = \sqrt{5}+1.$$

2.  $A=180^\circ - B - C=60^\circ$ ;

If  $AD$  be the perpendicular, then  $c=AD \operatorname{cosec} 45^\circ = 3\sqrt{2}$ ,

$$b=AD \operatorname{cosec} 75^\circ = \frac{6\sqrt{2}}{\sqrt{3}+1} = 3(\sqrt{6}-\sqrt{2})$$

$$a=BD+DC=AD \cot 45^\circ + AD \cot 75^\circ = 3 + 3(2-\sqrt{3}) = 9-3\sqrt{3}.$$

$$3. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{28 - 16\sqrt{3} + 24 - 12\sqrt{3} - 4}{4\sqrt{2}(9 - 5\sqrt{3})} = \frac{48 - 28\sqrt{3}}{4\sqrt{2}(9 - 5\sqrt{3})}$$

$$= \frac{(12 - 7\sqrt{3})(9 + 5\sqrt{3})}{\sqrt{2} \cdot 6} = \frac{3 - 3\sqrt{3}}{6\sqrt{2}} = -\frac{\sqrt{3} - 1}{2\sqrt{2}};$$

$$\therefore A = 180^\circ - 75^\circ = 105^\circ.$$

$$\sin C = \frac{c}{a} \sin A = \frac{3\sqrt{2} - \sqrt{6}}{2} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \cdot \frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{2} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ;$$

and

$$B = 180^\circ - 105^\circ - 60^\circ = 15^\circ.$$

4. We have

$$b - a = 2, \quad ab = 4;$$

$$\therefore b + a = 2\sqrt{5}, \text{ rejecting the negative sign,}$$

$$\therefore a = \sqrt{5} - 1, \quad b = \sqrt{5} + 1;$$

$$\therefore \sin B = \frac{b \sin A}{a} = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \cdot \frac{\sqrt{5} - 1}{4} = \frac{\sqrt{5} + 1}{4};$$

$$\therefore B = 54^\circ \text{ or } 126^\circ;$$

$$\therefore C = 108^\circ \text{ or } 36^\circ.$$

$$5. \sin c = \frac{c \sin B}{b} = 150 \cdot \frac{1}{2} \cdot \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ \text{ or } 120^\circ; \text{ both values being admissible since } b < c;$$

$$\therefore A = 90^\circ \text{ or } 30^\circ.$$

$$\text{In the first case, } a = \sin A \cdot \frac{b}{\sin B} = 50\sqrt{3} \times 2 = 100\sqrt{3}.$$

$$\text{In the second case, } a = b = 50\sqrt{3}.$$

$$\text{If } B = 30^\circ, \quad C = 150^\circ, \quad b = 75^\circ;$$

$$\sin C = \frac{150}{75} \times \sin 30^\circ = 1;$$

$$\therefore C = 90^\circ, \text{ and there is no ambiguity.}$$

$$6. \text{ We have at once from a figure, } \sin B = \frac{\sqrt{5} - 1}{4}; \text{ whence } B = 18^\circ;$$

$$\therefore C = 180^\circ - 36^\circ - 18^\circ = 126^\circ.$$

7. Let  $ABC$  be the triangle, and let  $B = 22\frac{1}{2}^\circ$ ,  $C = 112\frac{1}{2}^\circ$ , and let  $AD$  be the perpendicular from  $A$  on  $BC$ . Then  $A = 180^\circ - 22\frac{1}{2}^\circ - 112\frac{1}{2}^\circ = 45^\circ$ ; also

$$AD = AB \sin 22\frac{1}{2}^\circ,$$

$$BC = \frac{AB}{\sin 112\frac{1}{2}^\circ} \cdot \sin 45^\circ = \frac{AB}{\sqrt{2} \cos 22\frac{1}{2}^\circ} = \frac{2AB \sin 22\frac{1}{2}^\circ}{\sqrt{2} \sin 45^\circ} = 2AD.$$

That is, the altitude is half the base.

$$8. \frac{\sin A}{\sin B} = \frac{a}{b} = 2; \text{ and } \sin A = 2 \sin B = \sin 3B;$$

$$\therefore \frac{3 \sin B - 4 \sin^3 B}{\sin B} = 2, \text{ or } 3 - 4 \sin^2 B = 2.$$

$$\therefore \sin B = \frac{1}{2}, \text{ rejecting the negative value.}$$

$$\therefore B = 30^\circ, A = 90^\circ, C = 60^\circ.$$

Also  $c = a \sin C = \frac{a\sqrt{3}}{2}.$

9. Let  $C$  be the greatest angle, then

$$\begin{aligned} \cos C &= \frac{(2x+3)^2 + (x^2+2x)^2 - (x^2+3x+3)^2}{2(2x+3)(x^2+2x)} \\ &= \frac{-2x^3 - 7x^2 - 6x}{2(2x+3)(x^2+2x)} = -\frac{(2x+3)(x+2)}{2(2x+3)(x+2)} = -\frac{1}{2}. \end{aligned}$$

Thus the greatest angle is  $120^\circ$ .

$$10. \text{ First side} = b \cos C + c \cos B - a \cos C - c \cos A = a - b.$$

Now 
$$\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}}$$

$$= \frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}} = \sin \frac{A-B}{2} \operatorname{cosec} \frac{A+B}{2}.$$

$$11. \frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{\sin B + \sin(A+B)}{\sin A} = \frac{2 \sin \left(B + \frac{A}{2}\right) \cos \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}}.$$

$$\therefore (b+c) \sin \frac{A}{2} = a \sin \left(\frac{A}{2} + B\right).$$

$$12. \frac{a+b}{b+c} = \frac{\sin A + \sin B}{\sin B + \sin C} = \frac{\sin(B+C) + \sin B}{\sin B + \sin C} = \frac{2 \sin \left(B + \frac{C}{2}\right) \cos \frac{C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$= \frac{\sin \left(B + \frac{C}{2}\right) \cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B-C}{2}}.$$

$$13. \text{ First side} = \frac{1 - \cos(A - B) \cos(A + B)}{1 - \cos(A - C) \cos(A + C)} = \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)}$$

$$= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{a^2 + b^2}{a^2 + c^2}.$$

$$14. \text{ We have } c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0;$$

$$\therefore (c^2 - a^2 + ab + b^2)(c^2 - a^2 - ab + b^2) = 0;$$

$$\therefore c^2 = a^2 + ab + b^2, \text{ or } a^2 - ab + b^2;$$

But  $c^2 = a^2 + b^2 + 2ab \cos C;$

$$\therefore 2 \cos C = 1, \text{ or } -1;$$

$$\therefore C = 60^\circ, \text{ or } 120^\circ.$$

15. See figure of the Ambiguous Case on page 131.

$$(1) \quad c_1 - c_2 = B_1B_2 = 2B_1D = 2a \cos B_1.$$

$$(2) \quad \cos \frac{C_1 - C_2}{2} = \cos B_1CD = \frac{CD}{CB_1} = \frac{b \sin A}{a}.$$

$$(3) \quad c_1, c_2 \text{ are the roots of the quadratic}$$

$$c^2 - 2b \cos A \cdot c + b^2 - a^2 = 0; \quad [\text{Art. 150.}]$$

$$\therefore c_1 + c_2 = 2b \cos A; \quad c_1c_2 = b^2 - a^2.$$

$$\therefore c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = c_1^2 + c_2^2 - 2c_1c_2(2 \cos^2 A - 1)$$

$$= (c_1 + c_2)^2 - 4(b^2 - a^2) \cos^2 A$$

$$= 4b^2 \cos^2 A - 4b^2 \cos^2 A + 4a^2 \cos^2 A$$

$$= 4a^2 \cos^2 A.$$

$$(4) \text{ We have } \quad C_1 + C_2 = 2 \angle ACD;$$

$$C_1 - C_2 = 2 \angle B_1CD;$$

$$\therefore \sin \frac{C_1 + C_2}{2} \sin \frac{C_1 - C_2}{2} = \sin ACD \sin B_1CD$$

$$= \cos CAB_1 \cos CB_1A$$

$$= \cos A \cos B.$$

16. See figure of Art. 148 (iii). We have  $A = 45^\circ; \therefore \angle ACD = 45^\circ.$

Hence  $DC = DA = \frac{c_1 + c_2}{2};$  also  $DB_2 = \frac{c_1 - c_2}{2};$

$$\therefore CB_2^2 = \left(\frac{c_1 + c_2}{2}\right)^2 + \left(\frac{c_1 - c_2}{2}\right)^2 = \frac{c_1^2 + c_2^2}{2};$$

$$\therefore \cos^2 B_2CD = \frac{CD^2}{CB_2^2} = \frac{(c_1 + c_2)^2}{2(c_1^2 + c_2^2)};$$

$$\therefore \cos B_1CB_2 = 2 \cos^2 B_2CD - 1 = \frac{(c_1 + c_2)^2}{c_1^2 + c_2^2} - 1 = \frac{2c_1c_2}{c_1^2 + c_2^2}.$$

17. From the given condition we have

$$\sin C \cos A + 2 \cos C \sin C = \sin B \cos A + 2 \cos B \sin B,$$

or

$$\cos A (\sin C - \sin B) = \sin 2B - \sin 2C.$$

This easily reduces to

$$\cos A \sin \frac{B-C}{2} \cos \frac{B+C}{2} = 2 \cos A \sin \frac{B-C}{2} \cos \frac{B-C}{2}.$$

Now  $\cos \frac{B+C}{2}$  cannot equal  $2 \cos \frac{B-C}{2}$ ; hence we must have

$$\cos A = 0, \text{ which gives } A = 90^\circ;$$

or

$$\sin \frac{B-C}{2} = 0, \text{ in which case } B = C.$$

18. Since  $a, b, c$  are in A.P.; we have  $a - b = b - c$ ;

$$\therefore \sin A - \sin B = \sin B - \sin C;$$

$$\therefore 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2} = 2 \sin \frac{B-C}{2} \cos \frac{B+C}{2};$$

$$\therefore \frac{\sin \frac{A-B}{2}}{\sin \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin \frac{B-C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}};$$

$$\therefore \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2};$$

That is

$$\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2} \text{ are in A.P.}$$

19. Let  $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ ; then

$$\text{first side} = \frac{k^2 \sin A \sin B + \sin C \sin B - C}{\sin B + \sin C} + \dots + \dots$$

$$= k^2 \left[ \frac{\sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C} + \dots + \dots \right]$$

$$= k^2 [\sin A (\sin B - \sin C) + \dots + \dots] = 0.$$

### MISCELLANEOUS EXAMPLES. D. PAGE 138.

$$1. (1) \tan 2\theta \cot \theta - 1 = \frac{2}{1 - \tan^2 \theta} - 1 = \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \sec 2\theta.$$

$$(2) \sin \alpha - \cot \theta \cos \alpha = \frac{\sin \alpha \sin \theta - \cos \alpha \cos \theta}{\sin \theta} = -\operatorname{cosec} \theta \cos (\alpha + \theta).$$

$$2. \quad c^2 = a^2 + b^2 - 2ab \cos C = 48^2 + 35^2 - 48 \times 35 \\ = 13^2 + 48 \times 35 = 1849; \\ \therefore c = 43.$$

$$3. \quad \text{Here } \tan \alpha = \sqrt{\left(\frac{17}{8}\right)^2 - 1} = \frac{15}{8};$$

$$\text{also} \quad \tan \beta = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \frac{8}{15} = \cot \alpha;$$

$$\therefore \alpha + \beta = 90^\circ;$$

$$\therefore \tan(\alpha + \beta) = \infty, \text{ and } \operatorname{cosec}(\alpha + \beta) = 1.$$

$$4. \quad \text{The expression} = \frac{2 \sin 8a \cos 15a}{2 \sin 8a \cos 6a} = \frac{\cos 15a}{\cos 6a} = -1,$$

since  $15a = \pi - 6a$ .

$$5. \quad \text{First side} = \frac{1}{2} (\sin 3\theta - \sin \theta + \sin 5\theta - \sin 3\theta + \sin 7\theta - \sin 5\theta) \\ = \frac{1}{2} (\sin 7\theta - \sin \theta) \\ = \sin 3\theta \cos 4\theta.$$

$$6. \quad a^2 = 2 + 4 + 2\sqrt{3} - 2(\sqrt{3} + 1) = 4; \text{ whence } a = 2.$$

$$\text{Again} \quad \sin B = \frac{b \sin A}{a} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2};$$

$\therefore B = 30^\circ$ , or  $150^\circ$ ; but the latter value is inadmissible since  $c$  is the greatest side. Therefore  $C = 180^\circ - (45^\circ + 30^\circ) = 105^\circ$ .

$$7. \quad (1) \quad 2 \sin^2 36^\circ = 1 - \cos 72^\circ = 1 - \sin 18^\circ \\ = 1 - \frac{\sqrt{5} - 1}{4} = \frac{5 - \sqrt{5}}{4} = \sqrt{5} \sin 18^\circ.$$

$$(2) \quad 4 \sin 36^\circ \cos 18^\circ = 2 (\sin 54^\circ + \sin 18^\circ) \\ = \frac{\sqrt{5} + 1}{2} + \frac{\sqrt{5} - 1}{2} = \sqrt{5}.$$

$$8. \quad \frac{\sin 3a}{\sin a} + \frac{\cos 3a}{\cos a} = \frac{\sin 3a \cos a + \cos 3a \sin a}{\sin a \cos a} = \frac{2 \sin 4a}{\sin 2a} = 4 \cos 2a.$$

$$9. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 8 + 4\sqrt{3}}{8} = \frac{\sqrt{3}}{2};$$

$$\therefore A = 30^\circ.$$

$$\therefore B = C = \frac{1}{2} (180^\circ - 30^\circ) = 75^\circ.$$

10. (1) Each expression easily reduces to  $\frac{\sin a}{\sin 3a}$ .

$$\begin{aligned}
 (2) \quad \cos a + \cos 2a + \cos 3a &= 2 \cos \frac{3a}{2} \cos \frac{a}{2} + 2 \cos^2 \frac{3a}{2} - 1 \\
 &= 2 \cos \frac{3a}{2} \left( \cos \frac{a}{2} + \cos \frac{3a}{2} \right) - 1 \\
 &= 4 \cos a \cos \frac{a}{2} \cos \frac{3a}{2} - 1.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad (1) \quad \text{First side} &= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B \\
 &= 2b \sin C (b \cos C + c \cos B) \\
 &= 2ab \sin C = 2bc \sin A.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{First side} &= k (\sin A \sin \overline{B-C} + \dots + \dots) \\
 &= k (\sin B + C \sin B - C + \dots + \dots) \\
 &= k (\sin^2 B - \sin^2 C + \dots + \dots) = 0.
 \end{aligned}$$

$$12. \quad \tan(A+B) = \tan(360^\circ - \overline{C+D});$$

$$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = - \frac{\tan C + \tan D}{1 - \tan C \tan D}.$$

$$\begin{aligned}
 \therefore \tan A + \tan B + \tan C + \tan D &= \tan C \tan D (\tan A + \tan B) \\
 &\quad + \tan A \tan B (\tan C + \tan D);
 \end{aligned}$$

$$\begin{aligned}
 \text{or } \tan A + \tan B + \tan C + \tan D \\
 &= \tan A \tan B \tan C \tan D (\cot A + \cot B + \cot C + \cot D).
 \end{aligned}$$

### EXAMPLES. XIV. a. PAGE 145.

For Examples 1—3 see Arts. 151, 152.

For Examples 4—7 see Arts. 162, 163.

$$8. \quad \log 768 = \log (2^8 \times 3) = 8 \log 2 + \log 3 = 2.8853613.$$

$$9. \quad \log 2352 = \log (2^4 \times 3 \times 7^2) = 4 \log 2 + \log 3 + 2 \log 7 = 3.3714373.$$

$$\begin{aligned}
 10. \quad \log 35.28 &= \log \left( \frac{2^3 \times 3^2 \times 7^2}{10^2} \right) = 3 \log 2 + 2 (\log 3 + \log 7 - 1) \\
 &= .90309 + 2 (.4771213 + .845098 - 1) \\
 &= .90309 + .6444386 = 1.5475286.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \log \sqrt{6804} &= \frac{1}{2} \log (2^2 \times 3^5 \times 7) = \frac{1}{2} (2 \log 2 + 5 \log 3 + \log 7) \\
 &= \frac{1}{2} (.60206 + 2.3856065 + .845098) \\
 &= 1.9163822.
 \end{aligned}$$

$$12. \log \sqrt[5]{.00162} = \frac{1}{5} \log \left( \frac{2 \times 3^4}{10^3} \right) = \frac{1}{5} (\log 2 + 4 \log 3 - 5) \\ = \frac{1}{5} (\bar{3}.2095152) = \bar{1}.441903.$$

$$13. \log .0217 = \log \frac{217 - 21}{9000} = \log \frac{196}{9000} = \log \frac{7^2}{15^2 \times 10} \\ = 2 (\log 7 - \log 3 - \log 5) - 1 \\ = 2 (.8450980 - 1.1760913) - 1 = \bar{2}.3380134.$$

$$14. \log \cos 60^\circ = \log \left( \frac{1}{2} \right) = -\log 2 = -.30103 = \bar{1}.69897.$$

$$15. \log \sin^3 60^\circ = 3 \log \left( \frac{\sqrt{3}}{2} \right) = \frac{3}{2} \log 3 - 3 \log 2 \\ = .7156819 - .90309 = \bar{1}.8125919.$$

$$16. \log \sqrt[3]{\sec 45^\circ} = \frac{1}{3} \log \sqrt{2} = \frac{1}{6} \log 2 = .0501716.$$

$$17. \text{The expression} = \log \left[ \left( \frac{15}{8} \right)^2 \times \frac{162}{25} \times \left( \frac{4}{9} \right)^5 \right] \\ = \log \left( \frac{15 \times 15 \times 81 \times 2 \times 64}{64 \times 25 \times 81 \times 9} \right) = \log 2.$$

$$18. \text{The expression} \\ = 16 (1 - 2 \log 3) - 4 (2 \log 5 - 3 \log 2 - \log 3) - 7 (3 \log 2 + 1 - 4 \log 3) \\ = 9 - 9 \log 2 - 8 \log 5 = 1 - \log 2 = .69897.$$

$$19. \log x = \frac{1}{7} \log 7 = .1207283. \quad \therefore x = 1.320169.$$

$$20. \log x = \frac{1}{3} \log \left( \frac{2^2 \times 3^2 \times 7^2}{10^3} \right) = \frac{1}{3} (2 \log 2 + 2 \log 3 + 2 \log 7 - 8) \\ = \frac{1}{3} (3.2464986 - 8) = \bar{2}.4154995. \\ \therefore x = .0260315.$$

$$21. \log x = .5527899 + \bar{2}.5527899 + 1.1842633 = .2898431.$$

$$22. \log x = \frac{1}{3} \log \frac{11}{10^5} + 2 \log \frac{11^2}{10^2} + \frac{4}{3} \log \frac{11^3}{10^2} - \log (11^2 \times 10^5) \\ = \frac{1}{3} (\log 11 - 5) + 4 (\log 11 - 1) + \frac{4}{3} (3 \log 11 - 2) - 2 \log 11 - 5. \\ = \frac{19}{3} \log 11 - \frac{40}{3} = \frac{1}{3} (21.7864613) = \bar{7}.2621538.$$

23. Since  $\left(\frac{21}{20}\right)^{300} = \left(\frac{3 \times 7}{2^2 \times 5}\right)^{300}$ ;

$$\begin{aligned}\therefore \log \left(\frac{21}{20}\right)^{300} &= 300 (\log 3 + \log 7 - 1 - \log 2) \\ &= 396.66579 - 390.30900 = 6.35679;\end{aligned}$$

$\therefore \left(\frac{21}{20}\right)^{300}$  has 7 digits in its integral part.

Since  $\left(\frac{126}{125}\right)^{1000} = \left(\frac{2 \times 3^2 \times 7}{5^3}\right)^{1000}$ ;

$$\begin{aligned}\therefore \log \left(\frac{126}{125}\right)^{1000} &= 1000 (\log 2 + 2 \log 3 + \log 7 - 3 + 3 \log 2) \\ &= 1000 (4 \log 2 + 2 \log 3 + \log 7 - 3) \\ &= 3.4606;\end{aligned}$$

$\therefore \left(\frac{126}{125}\right)^{1000}$  has 4 digits in its integral part.

24.  $7^4$  is the smallest number whose logarithm has characteristic 4.

$7^3$  is the smallest number whose logarithm has characteristic 3.

$\therefore$  the required number is  $7^4 - 7^3 = 2058$ .

### EXAMPLES. XIV. b. PAGE 149.

$$\begin{aligned}1. \log x &= \frac{2}{3} \log \left(\frac{147 \times 75}{126 \times 16}\right) = \frac{2}{3} \log \left(\frac{7 \times 5^3}{2^5}\right) \\ &= \frac{2}{3} \log \frac{7 \times 1000}{16 \times 16} = \frac{2}{3} (3 + \log 7 - 8 \log 2) \\ &= \frac{2}{3} (3.8450980 - 2.4082400) = \frac{2}{3} (1.4368580) = .9579053;\end{aligned}$$

$\therefore x = 9.076226$ .

$$\begin{aligned}2. \log x &= \frac{1}{3} \log (3^3 \times 2 \times 7) + \frac{1}{2} \log (3^3 \times 2^2) \\ &\quad - \frac{1}{6} \log (3^2 \times 7 \times 2^4) - \frac{1}{3} \log (2 \times 3^3) \\ &= \frac{1}{2} \log 3 + \frac{1}{3} \log 2 + \frac{1}{6} \log 7 = .4797536;\end{aligned}$$

$\therefore x = 3.01824$ .

.1003433  
.2385606  
.1408497  
+ .4797536

$$\begin{aligned}
 3. \quad \log x &= \frac{1}{2} \log (10 \times 2^2 \times 3^3) + \frac{5}{3} \log (2^3 \times 3 \div 100) + \log (10 \times 3^4) \\
 &= \frac{43}{6} \log 3 + 6 \log 2 - \frac{11}{6} = \frac{1}{6} (20 \cdot 5162159 - 11) + 1 \cdot 8061800 \\
 &= 1 \cdot 5860360 + 1 \cdot 8061800 = 3 \cdot 3922160; \\
 \therefore x &= 2467 \cdot 266.
 \end{aligned}$$

$$4. \quad \text{Let } x \text{ be the value; then } 20^x = 800.$$

$$\therefore x \log 20 = \log 800;$$

$$\therefore x = \frac{\log 800}{\log 20} = \frac{2 + 3 \log 2}{1 + \log 2} = 2 \cdot 23.$$

$$5. \quad \text{Here } 3^x = 49, \text{ or } x \log 3 = 2 \log 7.$$

$$\therefore x = \frac{2 \log 7}{\log 3} = 3 \cdot 54.$$

$$6. \quad \text{Here } 125^x = 4000, \text{ or } x \log 5^3 = \log (2^2 \times 10^3).$$

$$\therefore x = \frac{3 + 2 \log 2}{3 \log 5} = 1 \cdot 72.$$

$$7. \quad \log x = \frac{40}{3} \log \left( \frac{378}{10^5} \right) = \frac{40}{3} (\log 2 + 3 \log 3 + \log 7 - 5) = 33 \cdot 699892.$$

$\therefore$  the number of ciphers is 32.

$$\log x = 50 \log \frac{259}{9990} = 50 \log \frac{7}{270}$$

$$= 50 (\log 7 - 3 \log 3 - 1) = 50 (0 \cdot 8450980 - 2 \cdot 4313639)$$

$$= 50 (2 \cdot 4137341) = 80 \cdot 6867050.$$

$\therefore$  the number of ciphers is 79.

$$8. \quad \text{Let } x \text{ be the base; then } x^3 = 11000;$$

$$\therefore 3 \log x = \log 11 + 3 = 4 \cdot 0413927;$$

$$\therefore \log x = 1 \cdot 3471309.$$

But

$$\log 222398 = 5 \cdot 3471309;$$

$$\therefore x = 22 \cdot 2398.$$

$$9. \quad \text{Here}$$

$$(x-1) \log 2 = \log 5 = 1 - \log 2.$$

$$\therefore x \log 2 = 1;$$

$$\therefore x = \frac{1}{\log 2} = \frac{1}{0 \cdot 30103} = 3 \cdot 32.$$

$$10. \quad \text{Here}$$

$$(x-4) \log 3 = \log 7.$$

$$\therefore x - 4 = \frac{\log 7}{\log 3} = 1 \cdot 77;$$

$$\therefore x = 5 \cdot 77.$$

11. Here  $(1-x) \log 5 = (x-3) (\log 2 + \log 3);$   
 $\therefore x (\log 2 + \log 3 + \log 5) = 3 \log 2 + 3 \log 3 + \log 5;$   
 $\therefore x = \frac{3.0334239}{1.4771213} = 2.05.$

12. Put  $\log 2 = a, \log 5 = b;$  then  
 $bx = -ay, b(2+y) = a(2-x).$

From these equations we obtain

$$x = \frac{2a}{a+b} = \frac{2 \log 2}{\log 2 + \log 5} = \frac{2 \log 2}{\log 10} = 2 \log 2 = .60206,$$

$$y = -\frac{ax}{b} = -2 \log 5 = -1.39794.$$

13. Put  $\log 2 = a, \log 3 = b;$  then  
 $ax = by, a(y+1) = b(x-1).$

From these equations we obtain

$$x = \frac{b}{b-a} = \frac{\log 3}{\log 3 - \log 2} = 2.71,$$

$$y = \frac{a}{b-a} = \frac{\log 2}{\log 3 - \log 2} = x - 1 = 1.71.$$

14. We have  $2 \log 2 + \log 7 = a, \log 3 + \log 7 = b, 2 - 2 \log 2 = c.$   
 $\therefore 2 + \log 7 = a + c, \text{ so that } \log 3 = b - a - c + 2.$   
 $\therefore \log 27 = 3(b - a - c + 2).$

Again  $\log 224 = \log (2^5 \times 7) = 5 \log 2 + \log 7.$   
 $= \frac{5}{2}(2 - c) + (a + c - 2) = \frac{1}{2}(2a - 3c + 6).$

15. We have  $\log (2 \times 11^2) = a, \log (2^3 \times 10) = b, \log (3^2 \times 5) = c.$   
 $\therefore 2 \log 11 + \log 2 = a, 3 \log 2 + 1 = b, 2 \log 3 + 1 - \log 2 = c.$

From the last two equations

$$b + c - 2 = 2 \log 3 + 2 \log 2 = \log 36.$$

Again,  $\log 66 = \log 6 + \log 11 = \frac{1}{2}(b + c - 2) + \frac{a - \log 2}{2}$   
 $= \frac{1}{2} \left[ b + c - 2 + a - \frac{1}{3}(b - 1) \right] = \frac{1}{6}(3a + 2b + 3c - 5).$

### MISCELLANEOUS EXAMPLES. E. PAGE 150.

1. First side  $= \frac{1}{2} (\cos 60^\circ + \cos 2A - \cos 120^\circ - \cos 2A)$   
 $= \frac{1}{2} (\cos 60^\circ - \cos 120^\circ) = \frac{1}{2}.$

2. See Ex. 1, p. 118, and Ex. 3, p. 119.

$$3. \quad b^2 = a^2 + c^2 - 2ac \cos B = 4 + 2 - 2 \cdot 2 \cdot \sqrt{2} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} = 4 - 2\sqrt{3} = (\sqrt{3}-1)^2.$$

$$\therefore b = \sqrt{3}-1.$$

$$\sin A = \frac{a}{b} \sin B = \frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$\therefore A = 45^\circ$ , or  $135^\circ$ ; but  $a$  is the greatest side, so that  $A = 135^\circ$ , and  $C = 30^\circ$ .

4. Each side easily reduces to 1.

$$5. \quad \text{Here } \frac{a}{b} = \frac{\cos A}{\cos B}; \text{ but in any triangle } \frac{a}{b} = \frac{\sin A}{\sin B}.$$

$$\therefore \frac{\sin A}{\sin B} = \frac{\cos A}{\cos B}, \text{ or } \sin(A-B) = 0;$$

$\therefore A = B$ , and the triangle is isosceles.

$$6. \quad (1) \text{ First side} = \frac{1}{2} (\cos 2\theta - \cos 4\theta + \cos 4\theta - \cos 6\theta + \cos 6\theta - \cos 8\theta + \cos 8\theta - \cos 10\theta)$$

$$= \frac{1}{2} (\cos 2\theta - \cos 10\theta) = \sin 6\theta \sin 4\theta.$$

$$(2) \text{ First side} = \frac{2 \sin 2a \cos a + 2 \sin 6a \cos a}{2 \cos 2a \cos a + 2 \cos 6a \cos a} \\ = \frac{\sin 2a + \sin 6a}{\cos 2a + \cos 6a} = \frac{2 \sin 4a \cos 2a}{2 \cos 4a \cos 2a} = \tan 4a.$$

$$7. \quad \text{First side} = \frac{\cos 3a \cos a + \sin 3a \sin a}{\sin a \cos a} = \frac{2 \cos (3a-a)}{2 \sin a \cos a} \\ = \frac{2 \cos 2a}{\sin 2a} = 2 \cot 2a.$$

$$8. \quad c^2 = a^2 + b^2 - 2ab \cos C = a^2 + (4 - 2\sqrt{3})a^2 - a^2(\sqrt{3}-1)\sqrt{3} = (2 - \sqrt{3})a^2.$$

$$\therefore a^2 = (2 + \sqrt{3})c^2, \text{ and } \sin^2 A = (2 + \sqrt{3}) \sin^2 C;$$

$$\therefore \sin^2 A = \frac{2 + \sqrt{3}}{4} = \frac{4 + 2\sqrt{3}}{8};$$

$$\therefore \sin A = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Hence  $A = 75^\circ$ , or  $105^\circ$ , and the latter value must be taken, as  $A = 75^\circ$  would make the triangle isosceles. Hence also  $B = 45^\circ$ .

$$9. \quad \tan 4a = \frac{2 \tan 2a}{1 - \tan^2 2a}; \text{ substitute } \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

$$\begin{aligned}
 10. (1) \text{ First side} &= a^2(1 - 2\sin^2 B) + b^2(1 - 2\sin^2 A) \\
 &= a^2 + b^2 - 2a^2\sin^2 B - 2b^2\sin^2 A = a^2 + b^2 - 4a^2\sin^2 B \\
 &= a^2 + b^2 - 4a\sin B \cdot b\sin A = a^2 + b^2 - 4ab\sin A\sin B.
 \end{aligned}$$

$$\begin{aligned}
 (2) \text{ First side} &= 2bc(1 + \cos A) + \dots + \dots \\
 &= (2bc + b^2 + c^2 - a^2) + \dots + \dots = (a + b + c)^2.
 \end{aligned}$$

$$\begin{aligned}
 11. \text{ From the equation } c^4 - 2c^2(a^2 + b^2) + (a^4 + b^4) &= 0 \text{ we have} \\
 \{c^2 - (a^2 + ab\sqrt{2} + b^2)\} \{c^2 - (a^2 - ab\sqrt{2} + b^2)\} &= 0.
 \end{aligned}$$

Equating the two factors separately to zero, we get

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}};$$

whence

$$C = 45^\circ \text{ or } 135^\circ.$$

$$12. \text{ We have } 2\cos\frac{3(A+B)}{2}\cos\frac{3(A-B)}{2} + \cos 3C = 1;$$

$$\therefore 2\cos\left(270^\circ - \frac{3C}{2}\right)\cos\frac{3(A-B)}{2} + \cos 3C = 1;$$

$$\therefore -2\sin\frac{3C}{2}\cos\frac{3(A-B)}{2} + 1 - 2\sin^2\frac{3C}{2} = 1;$$

$$\text{or } -2\sin\frac{3C}{2}\left[\cos\frac{3(A-B)}{2} + \sin\frac{3C}{2}\right] = 0;$$

$$2\sin\frac{3C}{2}\left[\cos\frac{3(A-B)}{2} - \cos\frac{3(A+B)}{2}\right] = 0;$$

$$4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2} = 0.$$

Since  $A, B, C$  are the angles of a triangle we must have one of the angles  $\frac{3A}{2}, \frac{3B}{2},$  or  $\frac{3C}{2}$  equal to  $180^\circ$ . That is, one of the angles of the triangle must be  $120^\circ$ .

### EXAMPLES. XV. a. PAGE 155.

$$\begin{aligned}
 1. \log 49517 &= 4.6947543 \\
 \log 49516 &= 4.6947456 \\
 \text{diff. for 1} &= \frac{87}{34}
 \end{aligned}$$

$$\begin{array}{r}
 348 \\
 261 \\
 \hline
 \text{diff. for } .34 = 29.58
 \end{array}$$

$$\log 49516 = 4.694756$$

$$\log 49516.34 = 4.694786$$

$$\therefore \log 4951634 = 6.694786.$$

$$\begin{aligned}
 2. \log 3.4714 &= .5405047 \\
 \log 3.4713 &= .5404921 \\
 \text{diff. for } .0001 &= \frac{126}{026}
 \end{aligned}$$

$$\begin{array}{r}
 756 \\
 252 \\
 \hline
 \text{diff. for } .0000026 = 3.276
 \end{array}$$

$$\log 3.4713 = .5404921$$

$$\log 3.4713026 = .5404924$$

$$\begin{array}{rcl} 3. \log 28497 & = & 4.4547991 \\ \log 28496 & = & 4.4547839 \\ \text{diff. for } 1 & = & \underline{152} \\ & & .14 \end{array}$$

$$\begin{array}{r} 6 \ 08 \\ 15 \ 2 \\ \hline \text{diff. for } .14 = 21 \ 28 \end{array}$$

$$\log 28496 = 4.4547839$$

$$\log 28496.14 = 4.4547860$$

$$\therefore \log 2849614 = 6.4547860.$$

$$\begin{array}{rcl} 5. \log 60814 & = & 1.7840036 \\ & & 43 \ 2 \\ & & \hline & & 3 \ 60 \end{array}$$

$$\therefore \log 6081465 = 6.7840083$$

$$7. \log x = 2.8283676$$

$$\log 67354 = 2.8283634$$

$$\text{diff.} = 42,$$

$$\text{and diff. for } 1 = 64;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{42}{64} = \frac{21}{32} = .66;$$

$$\therefore x = 673.5466.$$

$$9. \log x = 3.9184377$$

$$\log .0082877 = 3.9184340$$

$$\text{diff.} = 37,$$

$$\text{and diff. for } 1 = 52;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{37}{52} = .71;$$

$$\therefore x = .008287771.$$

$$11. \log x = \frac{1}{7} \log 142.71$$

$$= \frac{1}{7} (2.1544544)$$

$$= .3077792$$

$$\log 2.0313 = .3077741$$

$$\text{diff.} = 51,$$

$$\text{and diff. for } 1 = 213;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{51}{213} = .24;$$

$$\therefore x = 2.031324.$$

$$\begin{array}{rcl} 4. \log 57.634 & = & 1.7606788 \\ \log 57.633 & = & 1.7606712 \\ \text{diff. for } .001 & = & \underline{76} \\ & & .25 \end{array}$$

$$3 \ 80$$

$$15 \ 2$$

$$\text{diff. for } .00025 = 19 \ 00$$

$$\log 57.633 = 1.7606712$$

$$\log 57.63325 = 1.7606731$$

$$6. \log x = 4.7461735$$

$$\log 55740 = 4.7461670$$

$$\text{diff.} = 65,$$

$$\text{and diff. for } 1 = 78;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{65}{78} = \frac{5}{6} = .83;$$

$$\therefore x = 55740.8.$$

$$8. \log x = 2.0288435$$

$$\log .010686 = 2.0288152$$

$$\text{diff.} = 283,$$

$$\text{and diff. for } 1 = 406;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{283}{406} = .7;$$

$$\therefore x = .0106867.$$

$$10. \log x = 1.4034508$$

$$\log .25319 = 1.4034465$$

$$\text{diff.} = 43,$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{43}{172} = \frac{1}{4} = .25;$$

$$\therefore x = .2531925.$$

$$\begin{array}{rcl} 12. \log 13.894 & = & 1.1428273 \\ & & 281 \ 7 \\ & & \hline & & 6 \ 26 \end{array}$$

$$8 \ 1.1428561$$

$$\log x = .1428570$$

$$\log 1.3894 = .1428273$$

$$\text{diff.} = 297$$

$$\text{and diff. for } 1 = 313;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{297}{313} = .95;$$

$$\therefore x = 1.389495.$$

$$13. \log 24244 = \begin{array}{r} 5.3846043 \\ 125 \overline{) 3} \end{array}$$

$$14 \overline{) 5.3846168}$$

$$\log x = .3846155$$

$$\log 2.4244 = .3846043$$

$$\text{diff.} = \underline{112},$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{112}{179} = .63;$$

$$\therefore x = 2.424463.$$

$$14. \log 20691 = \begin{array}{r} 6.3157815 \\ 63 \overline{) 0} \\ 16 \overline{) 80} \end{array}$$

$$20 \overline{) 6.3157895}$$

$$\log x = .3157895$$

$$\log 2.0691 = .3157815$$

$$\text{diff.} = \underline{80},$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{80}{210} = \frac{8}{21} = .38;$$

$$\therefore x = 2.069138.$$

### EXAMPLES. XV. b. PAGE 159.

1.  $\sin 38^\circ 3' = .6163489;$  diff. for  $60'' = 2291.$

$$\text{prop}^l. \text{ increase} = \frac{35}{60} \times 2291 = \frac{1336}{.6164825}$$

3.  $\text{cosec } 55^\circ 21' = 1.2155978;$  diff. for  $60'' = 2443.$

$$\text{prop}^l. \text{ decrease} = \frac{28}{60} \times 2443 = \frac{1140}{1.2154838}$$

4.  $\sec \theta - \sec 62^\circ 42' = 6321;$

$$\text{diff. for } 60'' = 12296;$$

$$\text{and } \frac{6321}{12296} \times 60'' = 31'';$$

$$\therefore \theta = 62^\circ 42' 31''.$$

5.  $\cos 30^\circ 40' - \cos \theta = 560;$

$$\text{diff. for } 60'' = 1484;$$

$$\text{and } \frac{560}{1484} \times 60'' = 23'';$$

$$\therefore \theta = 30^\circ 40' 23''.$$

6.  $\cot 48^\circ 45' - \cot \theta = 3762;$

$$\text{diff. for } 60'' = 5145;$$

$$\text{and } \frac{3762}{5145} \times 60'' = 44'';$$

$$\therefore \theta = 48^\circ 45' 44''.$$

7.  $L \sin 44^\circ 17' = 9.8439842;$  diff. for  $60'' = 1295.$

$$\text{prop}^l. \text{ increase} = \frac{33}{60} \times 1295 = \frac{712}{9.8440554}$$

9.  $L \cos 55^\circ 30' = 9.7531280;$  diff. for  $60'' = 1838.$

$$\text{prop}^l. \text{ decrease} = \frac{24}{60} \times 1838 = \frac{735}{9.7530545}$$

$$\begin{aligned}
 10. \quad L \sin \theta - L \sin 44^\circ 17' &= 176; & 11. \quad L \cos 55^\circ 30' - L \cos \theta &= 1205; \\
 \text{diff. for } 60'' &= 1295; & \text{diff. for } 60'' &= 1838; \\
 \text{and } \frac{176}{1295} \times 60'' &= 8''; & \text{and } \frac{1205}{1838} \times 60'' &= 39''; \\
 \therefore \theta &= 44^\circ 17' 8''. & \therefore \theta &= 55^\circ 30' 39''.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad L \tan 24^\circ 50' &= 9.6653662; \quad \text{diff. for } 60'' = 3313. \\
 \text{prop}^l. \text{ increase} &= \frac{52.5}{60} \times 3313 = \frac{2899}{9.6656561}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{The required angle is } 42.5'' \text{ less than } 40^\circ 5'; \\
 \therefore \text{prop}^l. \text{ increase} &= \frac{42.5}{60} \times 1502 = 1064 \\
 L \operatorname{cosec} 40^\circ 5' &= 10.1911808 \\
 L \operatorname{cosec} 40^\circ 4' 17.5'' &= 10.1912872
 \end{aligned}$$

EXAMPLES. XV. c. PAGE 161.

$$\begin{array}{rcl}
 1. & \log 300.26 & = 2.4774975 \\
 & \begin{array}{r} 1 \\ 8 \end{array} & \begin{array}{r} 15 \\ 11 \end{array} 6 \\
 & \log .0078915 & = \bar{3}.8971596 \\
 & \begin{array}{r} 1 \\ 9 \\ 4 \end{array} & \begin{array}{r} 6 \\ 5 \\ 0 \end{array} 22 \\
 & & \hline
 & \log 2.3695 & = \begin{array}{r} .3746609 \\ .3746567 \end{array} \\
 & \begin{array}{r} 2 \end{array} & \begin{array}{r} 42 \\ 37 \end{array}
 \end{array}$$

Thus the product = 2.36952.

$$\begin{array}{rcl}
 2. & \log 235.67 & = 2.3723043 \\
 & \begin{array}{r} 8 \\ 3 \end{array} & \begin{array}{r} 148 \\ 5 \end{array} 6 \\
 & \log 357.84 & = 2.5536889 \\
 & \begin{array}{r} 3 \\ 8 \end{array} & \begin{array}{r} 36 \\ 9 \end{array} 8 \\
 & & \hline
 & \log 84336 & = \begin{array}{r} 4.9260131 \\ 4.9260130 \end{array}
 \end{array}$$

Thus the product is 84336.

3.

$$\begin{array}{rcl}
 \log 153.24 & = & 2.1853721 \\
 & & \begin{array}{r} 1 \\ 9 \end{array} \begin{array}{r} 28 \\ 25 \end{array} 6 \\
 \log 2.8632 & = & .4568517 \\
 & & \begin{array}{r} 5 \\ 0 \end{array} \begin{array}{r} 76 \\ 0 \end{array} \\
 & & \begin{array}{r} 3 \\ 4 \end{array} \begin{array}{r} 46 \\ 23 \end{array} \\
 \log .075836 & = & \bar{2}.8798754 \\
 & & \begin{array}{r} 4 \\ 6 \end{array} \begin{array}{r} 34 \\ 34 \end{array} \\
 \log 33.274 & = & 1.5221148 \\
 & & \begin{array}{r} 7 \\ 5 \end{array} \begin{array}{r} 98 \\ 91 \end{array} \\
 & & \begin{array}{r} 70 \\ 65 \end{array}
 \end{array}$$

Thus the product is 33.27475.

$$\begin{array}{rcl}
 4. \log 1.0304 & = & .0130059 \\
 & & \begin{array}{r} 0 \\ 5 \\ 1 \end{array} \begin{array}{r} 0 \\ 21 \\ 42 \end{array} \\
 & & \begin{array}{r} .0130081 \\ 1.4328656 \\ 2.5801425 \\ \hline 2.5801377 \end{array} \\
 \log .038031 & = & \bar{2}.5801377 \\
 & & \begin{array}{r} 4 \\ 2 \end{array} \begin{array}{r} 48 \\ 46 \end{array} \\
 & & \begin{array}{r} 20 \\ 23 \end{array}
 \end{array}$$

subtract

Thus the quotient is .03803142.

$$\begin{array}{rcl}
 \log 27.093 & = & 1.4328571 \\
 & & \begin{array}{r} 5 \\ 2 \\ 4 \end{array} \begin{array}{r} 81 \\ 32 \\ 64 \end{array} \\
 & & \begin{array}{r} 1.4328656 \end{array}
 \end{array}$$

$$\begin{array}{rcl}
 5. \log 357.83 & = & 2.5536767 \\
 & & \begin{array}{r} 6 \\ 4 \end{array} \begin{array}{r} 73 \\ 48 \end{array} \\
 & & \begin{array}{r} 2.5536845 \\ 3.5037539 \\ 5.0499306 \\ \hline 4.0499154 \end{array} \\
 \log 11218 & = & 4.0499154 \\
 & & \begin{array}{r} 4 \end{array} \begin{array}{r} 152 \\ 155 \end{array}
 \end{array}$$

$$\begin{array}{rcl}
 \log .0031897 & = & \bar{3}.5037498 \\
 & & \begin{array}{r} 3 \end{array} \begin{array}{r} 41 \end{array} \\
 & & \begin{array}{r} 3.5037539 \end{array}
 \end{array}$$

Thus the quotient is 112184.

$$\log .017834 = \frac{2.2512488}{5} = \frac{122}{2.2512610}$$

$$x = 1225.508.$$

$$\begin{array}{rcl}
 \log 3.7895 & = & .5785819 \\
 & 6 & 69 \\
 \log .053687 & = & \bar{2}.7298691 \\
 & 2 & 16 \\
 & & \hline
 & & 1.3084595 \\
 \log .0072916 & = & \bar{3}.8628228 \\
 & & \hline
 & & 1.4456367 \\
 \log 27.902 & = & 1.4456353 \\
 & 0 & 140 \\
 & 9 & 140
 \end{array}$$

8.  $\log .83410$   $= \bar{1}.9212181$

0  
3  
9

0  
1  
6  
47

---

1.9212183  
3

---

1.7636549

$\log .58030$   $= \bar{1}.7636526$

23  
22

3

Thus the cube is .580303.

9.

log 15063	= 4.1779115
0	0
1	29
8	230
	<hr/>
	5   4.1779120
	·8355824
log 6.8482	= ·8355764
	<hr/>
	60
	58
	<hr/>
	20
	19

Thus the fifth root is 6.848293.

$$\begin{array}{rcl}
 10. \log 384.73 & = & 2.5851561 \\
 & & \begin{array}{r} 11 \\ 5 \overline{) 2.5851572} \\ \cdot 5170314 \\ \hline \cdot 5170243 \end{array} \\
 \log 3.2887 & = & \cdot 5170243 \\
 & & \begin{array}{r} 71 \\ 5 \quad 66 \\ \hline 50 \\ 4 \quad 53 \end{array}
 \end{array}$$

Thus  $\sqrt[5]{384.73} = 3.288754$ .

$$\begin{array}{rcl}
 \log 15.732 & = & 1.1967839 \\
 & & \begin{array}{r} 111 \\ 13 \overline{) 1.1967950} \\ \cdot 0920612 \\ \hline \cdot 0920536 \end{array} \\
 \log 1.2361 & & \begin{array}{r} 76 \\ 2 \quad 70 \\ \hline 60 \\ 2 \quad 70 \end{array}
 \end{array}$$

Thus  $\sqrt[13]{15.7324} = 1.236122$ .

$$\begin{array}{rcl}
 11. \log 1034.3 & = & 3.0146465 \\
 & & \begin{array}{r} 379 \\ 6 \quad 25 \\ 3 \quad 1 \end{array} \\
 & & \begin{array}{r} 2 \overline{) 3.0146871} \\ 1.5073435 \\ \hline 1.8493591 \\ 3.3567026 \\ \hline 3.3566950 \end{array} \\
 \text{add} & & \\
 \log 2273.5 & = & 3.3566950 \\
 & & \begin{array}{r} 76 \\ 4 \quad 76 \end{array}
 \end{array}$$

$$\begin{array}{rcl}
 \log 35324 & = & 5.5480699 \\
 & & \begin{array}{r} 74 \\ 3 \overline{) 5.5480773} \\ \hline 1.8493591 \end{array}
 \end{array}$$

Thus the product is 2273.54.

12. Let  $a = 1.0356270$  and  $b = .7503269$ ; then  $a^2 - b^2 = (a + b)(a - b)$ , and  $a + b = 1.7859539$ ,  $a - b = .2853001$ .

$$\begin{array}{rcl}
 \log 1.7859 & = & .2518571 \\
 & & \begin{array}{r} 122 \\ 5 \quad 7 \\ 3 \quad 3 \\ 9 \quad 2 \end{array} \\
 \log .28530 & = & \bar{1}.4553018 \\
 & & \begin{array}{r} 0 \quad 0 \\ 1 \quad 1 \end{array} \\
 \log .50953 & & \begin{array}{r} 1.7071722 \\ \hline 1.7071698 \end{array} \\
 & & \begin{array}{r} 24 \\ 2 \quad 17 \\ \hline 70 \\ 8 \quad 69 \end{array}
 \end{array}$$

Thus the difference is .5095328.

$$13. \log x = \frac{3}{5} \log 34.7326 + \frac{1}{6} \log 2.53894 - \frac{1}{5} \log 4.39682.$$

$$\log 2.5389 \quad \begin{array}{r} 4 \\ 68 \end{array} = .4046456$$

$$\begin{array}{r} 6 \overline{) .4046524} \\ \underline{.0674421} \\ .7958146 \end{array}$$

add

$$\begin{array}{r} \underline{.8632567} \\ .8632515 \end{array}$$

$$\log 7.2988 \quad \begin{array}{r} 52 \\ 9 \quad 54 \end{array} =$$

Thus  $x = 7.29889$ .

$$\log 4.3968 \quad \begin{array}{r} 2 \\ 20 \end{array} = .6431367$$

$$\begin{array}{r} \underline{.6431387} \end{array}$$

$$\log 34.732 \quad \begin{array}{r} 6 \\ 75 \end{array} = 1.5407298$$

$$\begin{array}{r} \underline{1.5407373} \\ 3 \end{array}$$

$$\begin{array}{r} 4.6222119 \\ \underline{.6431387} \end{array}$$

$$\begin{array}{r} 5 \overline{) 3.9790732} \\ \underline{.7958146} \end{array}$$

$$\text{subtract}$$

$$\begin{array}{r} 5 \overline{) 3.9790732} \\ \underline{.7958146} \end{array}$$

$$\begin{array}{r} \underline{.7958146} \end{array}$$

$$14. \log .0037258 \quad \begin{array}{r} 1 \\ 6 \\ 9 \end{array} = \begin{array}{r} 3.5712195 \\ 12 \\ 70 \\ 105 \end{array}$$

add

$$\begin{array}{r} 3.5712215 \\ \underline{1.7505167} \end{array}$$

$$2 \overline{) 3.3217382}$$

$$\begin{array}{r} \underline{2.6608691} \\ 2.6608655 \end{array}$$

$$\log .045800 \quad \begin{array}{r} 36 \\ 29 \end{array} =$$

$$\begin{array}{r} 70 \\ 67 \end{array}$$

$$\begin{array}{r} 7 \end{array}$$

Thus the mean proportional is .04580037.

$$15. \text{ If } x \text{ be the required number, we have } x = \frac{.03751786}{(.43607528)^2}$$

$$\log .037517 \quad \begin{array}{r} 8 \\ 6 \end{array} = \begin{array}{r} 2.5742281 \\ 93 \\ 70 \end{array}$$

$$\begin{array}{r} \underline{2.5742381} \\ 1.2791230 \end{array}$$

$$\begin{array}{r} \underline{1.2951151} \\ 1.2951051 \end{array}$$

$$\log .19729 \quad \begin{array}{r} 100 \\ 88 \end{array}$$

$$\begin{array}{r} \underline{120} \\ 110 \end{array}$$

$$\begin{array}{r} 4 \end{array}$$

$$\begin{array}{r} 5 \end{array}$$

$$\begin{array}{r} 110 \end{array}$$

$$\begin{array}{r} 110 \end{array}$$

$$\begin{array}{r} 110 \end{array}$$

$$\begin{array}{r} 110 \end{array}$$

$$\begin{array}{r} 110 \end{array}$$

$$\begin{array}{r} 110 \end{array}$$

$$\log .43607 \quad \begin{array}{r} 5 \\ 2 \\ 8 \end{array} = \begin{array}{r} 1.6395562 \\ 50 \\ 20 \end{array}$$

$$\begin{array}{r} \underline{1.6395615} \\ 2 \end{array}$$

$$\begin{array}{r} \underline{1.2791230} \end{array}$$

Thus  $x = .1972945$ .

16. If  $x$  be the required number, we have

$$x = \frac{29.302564 \times .33025107}{56712.43}$$

log 29.302	= 1.4668973		log 56712	= 4.7536750
5	74		4	31
6	89		3	23
4	59			<u>4.7536783</u>
log .33025	= 1.5188428			
1	13			
0	0			
7	92			
	<u>.9857498</u>			
subtract	4.7536783			
	<u>4.2320715</u>			
log .00017063	= 4.2320554			
	161			
6	152			
	90			
3	76			
	<u>—</u>			

Thus the fourth proportional is .0001706363.

17. Let  $x$  be the required number, then

$$x = \sqrt{(.035689)^{\frac{2}{3}} \times (2.879432)^{\frac{3}{7}}}$$

log 2.8794	= .4593020		$\frac{1}{5} \log .035689 = \frac{1}{5} (2.5525344)$
3	45		$= 1.7105069$
2	30		
	<u>.4593068</u>		
	3		
14	1.3779204		
	<u>.0984229</u>		
	1.7105069		
	<u>1.8089298</u>		
log .64406	= 1.8089263		
	35		
5	34		
	<u>—</u>		

Thus the geometric mean is .644065.

18. Here  $x = \frac{(7836.43)^{\frac{1}{3}} \times (357.814)^{\frac{1}{3}}}{(32.7812)^{\frac{1}{3}}}$

$$\log 7836.4 = 3.8941166$$

$$3 \quad 17$$

$$4 \mid 3.8941183$$

$$\cdot 9735295$$

$$\cdot 5107315$$

$$1.4842610$$

subtract  $\cdot 5052083$

$$\cdot 9790527$$

$$\log 9.5291 = \cdot 9790519$$

$$8$$

$$2 \quad 9$$

$$\log 32.781 = 1.5156222$$

$$2 \quad 26$$

$$3 \mid 1.5156248$$

$$\cdot 5052083$$

$$\log 357.81 = 2.5536525$$

$$4 \quad 49$$

$$5 \mid 2.5536574$$

$$\cdot 5107315$$

Thus the fourth proportional is 9.52912.

19.  $\log \sin 27^\circ 13' = 1.6602550$

$$\frac{12}{60} \times 2455 = 491$$

$$1.6603041$$

$$1.8414768$$

$$1.5017809$$

$$\log .31752 = 1.5017711$$

$$98$$

$$7 \quad 96$$

$$20$$

$$1 \quad 14$$

Thus the required value is .3175271.

$$\log \cos 46^\circ 2' = 1.8415095$$

$$\text{subtract } \frac{15}{60} \times 1310 = 327$$

$$1.8414768$$

20.

$$\cot 97^\circ 14' 16'' = -\cot 82^\circ 45' 44'',$$

$$\sec 112^\circ 13' 5'' = -\sec 67^\circ 46' 55''.$$

$$\log \sec 67^\circ 46' = .4220725$$

$$\frac{11}{12} \times 3092 = 2834$$

$$\cdot 4223559$$

$$1.1038011$$

$$1.5261570$$

$$\log .33585 = 1.5261454$$

$$116$$

$$9 \quad 116$$

Thus the required value is .335859.

$$\log \cot 82^\circ 45' = 1.1045420$$

$$\frac{11}{15} \times 10103 = 7409$$

$$1.1038011$$

$$\begin{array}{rcl}
 21. \quad \log \sin 20^\circ 13' & = & \bar{1}.5385375 \\
 \frac{20}{60} \times 3429 & = & 1143 \\
 \log \sec 42^\circ 15' & = & .1306403 \\
 \frac{30}{60} \times 1148 & = & 574 \\
 & & \hline
 & & \bar{1}.6693495
 \end{array}$$

$$\begin{array}{rcl}
 \log \cot 47^\circ 53' & = & \bar{1}.9562154 \\
 \text{subtract } \frac{15}{60} \times 2540 & = & 635 \\
 & & \hline
 & & \bar{1}.9561519 \\
 \text{add} & & \bar{1}.6693495 \\
 & & \hline
 & & \bar{1}.6255014 \\
 \log .42218 & = & \bar{1}.6254977 \\
 & & \hline
 & & 37 \\
 & & 3 \quad 31 \\
 & & \hline
 & & 60 \\
 & & 6 \quad 62 \\
 & & \hline
 & & 62
 \end{array}$$

Thus the required value is .4221836.

$$\begin{array}{rcl}
 22. \quad \log 324.13 & = & 2.5107192 \\
 & 6 & 80 \\
 & 8 & 10 \quad 7 \\
 \log 417.24 & = & 2.6203859 \\
 & 3 & 31 \\
 & 1 & 10 \\
 & & \hline
 & & 5.1311174 \\
 & & \bar{1}.9632566 \\
 & & \hline
 & & 5.0943740 \\
 \log 12427 & = & 5.0943663 \\
 & & \hline
 & & 77 \\
 & 2 & 70 \\
 & & \hline
 & & 70 \\
 & 2 & 70 \\
 & & \hline
 & & 70
 \end{array}$$

$$\begin{array}{rcl}
 \log \sin 113^\circ 14' 16'' & & \\
 = \log \sin 66^\circ 45' 44'', & & \\
 \log \sin 66^\circ 45' & = & \bar{1}.9632168 \\
 \frac{44}{60} \times 543 & = & 398 \\
 & & \hline
 & & \bar{1}.9632566
 \end{array}$$

Thus the required value is 12427.2.

$$23. \quad \text{Here } a = \frac{b \sin A}{\sin B}, \text{ and } \sin B = \sin 60^\circ 45' 42''.$$

$$\begin{array}{rcl}
 \log \sin 35^\circ 15' & = & 1.7612851 \\
 \frac{33}{60} \times 1787 & = & 983 \\
 & & \hline
 & & \bar{1}.7613834 \\
 & & \bar{1}.9408130 \\
 & & \hline
 & & 1.8205704 \\
 \log 378.25 & = & 2.5777789 \\
 & & \hline
 & & 2.3983493 \\
 \log 250.23 & = & 2.3983394 \\
 & & \hline
 & & 99 \\
 & 5 & 87 \\
 & & \hline
 & & 120 \\
 & 7 & 121 \\
 & & \hline
 & & 121
 \end{array}$$

$$\begin{array}{rcl}
 \log \sin 60^\circ 45' & = & 1.9407634 \\
 \frac{42}{60} \times 708 & = & 496 \\
 & & \hline
 & & \bar{1}.9408130
 \end{array}$$

Thus  $a = 250.2357$ .

$$24. (1) \log \tan \theta = \frac{1}{3} (\log 5 - \log 12)$$

$$\log 5 = .6989700$$

$$\log 12 = 1.0791812$$

$$3 \overline{) 1.6197888}$$

$$\log \tan \theta = \overline{1.8732629}$$

$$\log \tan 36^\circ 45' = \overline{1.8731668}$$

$$961$$

$$\frac{961}{2634} \times 60'' = 22''.$$

$$\therefore \theta = 36^\circ 45' 22''.$$

$$(2) (3 \sin \theta - 1)(\sin \theta + 1) = 0.$$

$$\therefore \sin \theta = \frac{1}{3} \text{ or } -1.$$

$$\log \sin \theta = -\log 3$$

$$= \overline{1.5228787}$$

$$\log \sin 19^\circ 28' = \overline{1.5227811}$$

$$976$$

$$\frac{976}{3572} \times 60'' = 16''.$$

$$\therefore \theta = 19^\circ 28' 16''.$$

$$25. x = \sin 23^\circ 18' 5'' \times \cot 38^\circ 15' 13'' \times \cos 28^\circ 17' 25''.$$

$$\log \cot 38^\circ 15' = .1032884$$

$$\text{subtract } \frac{13}{60} \times 2598 = \frac{562}{.1032322}$$

$$\overline{1.9447579}$$

$$\log \sin 23^\circ 18' = \overline{1.5971965}$$

$$\frac{5}{60} \times 2932 = \frac{244}{\overline{1.6452110}}$$

$$\log .44178 = \overline{1.6452061}$$

$$49$$

$$5 \quad \underline{49}$$

$$\log \cos 28^\circ 17' = \overline{1.9447862}$$

$$\text{subtract } \frac{25}{60} \times 680 = \frac{283}{\overline{1.9447579}}$$

Thus  $x = .441785$ .

26.

$$\log \cos 32^\circ 47' = \overline{1.9246535}$$

$$2$$

$$\log \cot 41^\circ 19' = \overline{1.8493070}$$

$$3 \overline{) 1.9052998}$$

$$\overline{1.9684333}$$

$$\log \sin 68^\circ 25' = \overline{1.9684286}$$

$$47$$

$$\frac{47}{499} \times 60'' = 6''.$$

Thus  $\theta = 68^\circ 25' 6''$ .

## EXAMPLES. XV. d. PAGE 163 c.

1.  $\log 2834 = 3.4524$   
 $\log 17.62 = 1.2460$   
 $\log x = 4.6984$ ;  
whence  $x = 49940$ .
2.  $\log 8.034 = .9049$   
 $\log 1893 = 3.2772$   
 $\log x = 4.1821$ ;  
whence  $x = 15210$ .
3.  $\log .00567 = \bar{3}.7536$   
 $\log .0397 = \bar{2}.4728$   
 $\log x = 4.2264$ ;  
whence  $x = .0001685$
4.  $\log 3.7 = .5682$   
 $\log 8.9 = .9494$   
 $\log .023 = \bar{2}.3617$   
 $\log x = \bar{1}.8793$ ;  
whence  $x = .7573$ .
5.  $\log 31.9 = 1.5038$   
 $\log 1.51 = .1790$   
 $\log 9.7 = .9868$   
 $\log x = 2.6696$ ;  
whence  $x = 467.3$ .
6.  $\log 43 = 1.6335$   
 $\log 8.07 = .9069$   
 $\log .0392 = \bar{2}.5933$   
 $\log x = \bar{1}.1337$ ;  
whence  $x = 13.60$ .
7.  $\log 17.3 = 1.2380$   
 $\log 294.8 = 2.4695$   
 $\log x = \bar{2}.7685$ ;  
whence  $x = .05868$ .
8.  $\log 2.035 = .3086$   
 $\log 837.6 = 2.9230$   
 $\log x = \bar{3}.3856$ ;  
whence  $x = .00243$ .
9.  $\log .2179 = \bar{1}.3383$   
 $\log .08973 = \bar{2}.9529$   
 $\log x = .3854$ ;  
whence  $x = 2.429$ .
10.  $\log 487 = 2.6875$   
 $\log 6398 = 3.8060$   
 $\log x = \bar{2}.8815$ ;  
whence  $x = .07612$ .
11.  $\log 2.38 = .3766$   
 $\log 3.901 = .5912$   
 $\log 4.83 = .6839$   
 $\log x = .2839$ ;  
whence  $x = 1.923$ .
12.  $\log 14.72 = 1.1679$   
 $\log 38.05 = 1.5804$   
 $\log 387.9 = 2.5887$   
 $\log x = .1596$ ;  
whence  $x = 1.444$ .
13.  $\log 925.9 = 2.9665$   
 $\log 1.597 = .2034$   
 $\log 74.03 = 1.8694$   
 $\log x = \bar{3}.1699$   
 $\log x = \bar{1}.3005$ ; whence  $x = 19.97$ .
14.  $\log 15.38 = 1.1869$   
 $\log .0137 = \bar{2}.1367$   
 $\log 276 = 2.4409$   
 $\log .0038 = \bar{3}.5798$   
 $\log x = \bar{1}.3236$   
 $\log x = \bar{1}.3029$ ; whence  $x = .2008$ .

15.  $\log 2.31 = .3636$   
 $\log .037 = \bar{2}.5682$   
 $\log 1.43 = .1553$   
 $\quad \quad \quad \bar{1}.0871$   
 $\quad \quad \quad \bar{3}.2957$   
 $\log x = \bar{1}.7914$ ; whence  $x = 61.86$ .
16.  $\log x = \frac{1}{2} \log 5.1 = \frac{1}{2} (.7076) = .3538$ ; whence  $x = 2.258$ .
17.  $\log x = \frac{1}{3} \log 11 = \frac{1}{3} (1.0414) = .3471$ ; whence  $x = 2.224$ .
18.  $\log x = \frac{1}{3} \log 82.56 = \frac{1}{3} (1.9168) = .6839$ ; whence  $x = 4.354$ .
19.  $\log x = \frac{1}{4} \log 10.15 = \frac{1}{4} (1.0064) = .2516$ ; whence  $x = 1.781$ .
20.  $\log x = 4 \log .097 = 4 (\bar{2}.9868) = \bar{5}.9472$ ; whence  $x = .00008855$ .
21.  $5 \log 2.301 = .3619 \times 5 = 1.8095$ ; whence  $x = 64.49$ .
22.  $\frac{2}{3} \log 51.32 = \frac{1.7103 \times 2}{3} = 1.1402$ ; whence  $x = 13.81$ .
23.  $\frac{4}{7} \log .089 = \frac{\bar{2}.9494 \times 4}{7} = \bar{1}.3997$ ; whence  $x = .2510$ .
24.  $\log .0137 = \bar{2}.1367$   
 $\log .0296 = \bar{2}.4713$   
 $\quad \quad \quad = \bar{4}.6080$   
 $\log 873.5 = 2.9412$   
 $\quad \quad \quad 2) \bar{7}.6668$   
 $\quad \quad \quad \bar{4}.8334$ ; whence  $x = .0006814$ .
25.  $\log 83 = 1.9191$   
 $\frac{1}{3} \log 92 = .6546$   
 $\quad \quad \quad 2.5737$   
 $\quad \quad \quad 2.5820$   
 $\log x = \bar{1}.9917$ ; whence  $x = .9811$ .
26.  $\log .678 = \bar{1}.8312$   
 $\log 9.01 = .9547$   
 $\quad \quad \quad .7859$   
 $\log .0234 = \bar{2}.3692$   
 $\quad \quad \quad 2) 2.4167$   
 $\log x = 1.2084 = \log 16.15$ ;  
 $\therefore x = 16$ , to nearest integer.
- $\log .0561 = \bar{2}.7490$   
 $\log 3.87 = .5877$   
 $\log .0091 = \bar{3}.9590$   
 $\quad \quad \quad \bar{3}.2957$
- $\log 127 = 2.1038$   
 $\frac{1}{5} \log 246 = .4782$   
 $\quad \quad \quad 2.5820$

27. (i) If  $x$  is the mean proportional between 2.87 and 30.08,

$$x = \sqrt{2.87 \times 30.08}$$

$$\log 2.87 = .4579$$

$$\log 30.08 = 1.4782$$

$$\begin{array}{r} 2) 1.9361 \\ \hline \end{array}$$

$$\log x = .9680; \text{ whence } x = 9.29.$$

- (ii) If  $x$  is the third proportional to .0238 and 7.805

$$x \times .0238 = (7.805)^2; \therefore x = \frac{(7.805)^2}{.0238}.$$

$$2 \log 7.805 = 1.7848$$

$$\log .0238 = \bar{2}.3766$$

$$\log x = 3.4082; \text{ whence } x = 2560.$$

28. Here  $x = \{\sqrt[2]{347.3} \times \sqrt[3]{256.4}\}^{\frac{1}{2}}$   
 $= (347.3)^{\frac{1}{4}} \times (256.4)^{\frac{1}{6}}.$

$$\frac{1}{4} \log 347.3 = .4234$$

$$\frac{1}{6} \log 256.4 = .2409$$

$$\log x = .6643; \text{ whence } x = 4.616.$$

29.  $5 \log x + 3 \log y = \log 5,$   
 $2 \log x + 7 \log y = \log 11.$

These equations give

$$\log x = \frac{7 \log 5 - 3 \log 11}{29}, \quad \log y = \frac{5 \log 11 - 2 \log 5}{29}.$$

$$7 \log 5 = 4.8930$$

$$3 \log 11 = 3.1242$$

$$\begin{array}{r} 29) 1.7688 \\ \hline \end{array}$$

$$288$$

$$27$$

$$\therefore \log x = .0610;$$

$$\text{whence } x = 1.151.$$

$$5 \log 11 = 5.2070$$

$$2 \log 5 = 1.3980$$

$$\begin{array}{r} 29) 3.8090 \\ \hline \end{array}$$

$$90$$

$$39$$

$$100$$

$$\therefore \log y = .1313;$$

$$\text{whence } y = 1.353.$$

- 30.

$$\log l = \log 2.863 = .4569$$

$$\log g = \log 32.19 = 1.5077$$

$$\begin{array}{r} 2) 2.9492 \\ \hline \end{array}$$

$$\log \sqrt{\frac{l}{g}} = 1.4746$$

$$\log 2 = .3010$$

$$\log \pi = .4972$$

$$\begin{array}{r} \hline .2728 \\ \hline \end{array}$$

$$\text{whence the required value} = 1.874.$$

## THE USE OF LOGARITHMIC TABLES.

xv.]

31.

$$\log m = \log 18.34 = 1.2634$$

$$\log v^2 = 2 \log 35.28 = 3.0950$$

$$\underline{4.3584}$$

$$\log 2 = .3010$$

$$\log \frac{1}{2} mv^2 = 4.0574; \text{ whence } \frac{1}{2} mv^2 = 11410.$$

32. (i)

$$\log p = \log 93.75 = 1.9719$$

$$\log r^n = 4 \log 1.03 = .0512$$

$$\log pr^n = 2.0231; \text{ whence } pr^n = 105.4.$$

(ii)

$$\log r^3 = 3 \log 5.875 = 2.3070$$

$$\log \pi = .4971$$

$$\log 4 = .6021$$

$$\underline{3.4062}$$

$$\log 3 = .4771$$

$$\log \frac{4}{3} \pi r^3 = 2.9291; \text{ whence } \frac{4}{3} \pi r^3 = 849.4.$$

$$\log 355 = 2.5502$$

$$\log 113 = 2.0531$$

$$\log \pi = .4971$$

33.

$$\log m = \log 33.47 = 1.5246$$

$$\log v^2 = \log 3600 = 3.5563$$

$$\underline{5.0809}$$

$$\underline{2.4900}$$

$$\log F = 2.5909; \text{ whence } F = 389.8.$$

$$\log g = \log 32.19 = 1.5077$$

$$\log r = \log 9.6 = .9823$$

$$\underline{2.4900}$$

34.

$$r^3 = \frac{3 \times 537.6}{4 \times 3.1416}$$

$$\log 3 = .4771$$

$$\log 537.6 = 2.7305$$

$$\underline{3.2076}$$

$$\underline{1.0992}$$

$$3) 2.1084$$

$$\log r = .7028; \text{ whence } r = 5.044.$$

$$\log 4 = .6021$$

$$\log 3.1416 = .4971$$

$$\underline{1.0992}$$

35.

$$f = \frac{2s}{t^2} = \frac{578.6 \times 8^2}{31^2}$$

$$\log 578.6 = 2.7624$$

$$\log 64 = 1.8062$$

$$\underline{4.5686}$$

$$2 \log 31 = 2.9828$$

$$\log f = 1.5858; \text{ whence } f = 38.53.$$

36.

$$n \log x + \log y = 8 + \log 8.7$$

$$n \log 73.96 + \log 27.25 = 8 + \log 8.7 = 8.9395$$

$$\log 27.25 = 1.4354$$

$$\therefore n \log 73.96 = 7.5041.$$

$$\therefore n = \frac{7.5041}{\log 73.96} = \frac{7.5041}{1.8690}$$

$$= 4.015.$$

$$18.69) 75041 (4.015$$

$$\underline{281}$$

$$\underline{94}$$

$$\underline{1}$$

$$37. \quad r^3 = \frac{3V}{4\pi} = \frac{2 \times 33.87}{4 \times 3.1416}.$$

$$\begin{array}{r} \log 3 = .4771 \\ \log 33.87 = 1.5298 \\ \hline 2.0069 \\ 1.0992 \\ \hline 3) .9077 \\ \log r = .3026; \text{ whence } r = 2.007. \end{array} \quad \begin{array}{r} \log 4 = .6021 \\ \log 3.1416 = .4971 \\ \hline 1.0992 \end{array}$$

38. Let  $d$  be the diameter; then

$$\frac{4}{3} \pi \left(\frac{d}{2}\right)^3 = (36.4)^3;$$

$$\therefore d^3 = \frac{6 \times (36.4)^3}{\pi};$$

$$\therefore 3 \log d = \log 6 + 3 \log 36.4 - \log \pi.$$

Thus  $d = 45.16$  cm.

$$\begin{array}{r} \log 6 = .7782 \\ 3 \log 36.4 = 4.6833 \\ \hline 5.4615 \\ \log \pi = .4971 \\ \hline 3) 4.9644 = 3 \log d \\ \hline 1.6548 \\ \text{antilog } 1.6548 = 45.16. \end{array}$$

$$39. \quad \begin{aligned} 2 \log v &= \log r + \log g - \log 289 \\ &= \log 4000 + \log 32.2 - \log 5280 - \log 289. \end{aligned}$$

$$\begin{array}{r} \log 4000 = 3.6021 \\ \log 32.2 = 1.5079 \\ \hline 5.1100 \\ 6.1835 \\ \hline 2) 2.9265 \end{array} \quad \begin{array}{r} \log 5280 = 3.7226 \\ \log 289 = 2.4609 \\ \hline 6.1835 \end{array}$$

$$\log v = 1.4632; \text{ whence } v = .2905.$$

Let  $E = \frac{2\pi r}{v \times 60^2}$ ; then  $\log E = \log 2\pi r - \log v - 2 \log 60$

$$\begin{array}{r} \log 2 = .3010 \\ \log \pi = .4971 \\ \log r = 3.6021 \\ \hline 4.4002 \\ 3.0196 \\ \hline \log E = 1.3806; \text{ whence } E = 24, \text{ approximately.} \end{array} \quad \begin{array}{r} \log v = 1.4632 \\ 2 \log 60 = 3.5564 \\ \hline 3.0196 \end{array}$$

### EXAMPLES. XV. e. PAGE 163 H.

In Examples 16—19 let the expression be denoted by  $x$ .

$$16. \quad \begin{aligned} \log \sin 27^\circ 13' &= 1.6602 \\ \log \cos 46^\circ 16' &= 1.8397 \\ \hline \log x &= 1.4999; \\ \text{whence } x &= .3161. \end{aligned}$$

$$17. \quad \begin{aligned} \log \sin 47^\circ 13' &= 1.8656 \\ \log \tan 22^\circ 27' &= 1.6162 \\ \hline \log x &= .2494; \\ \text{whence } x &= 1.776. \end{aligned}$$

$$18. \quad \begin{aligned} \log \sin 34^\circ 17' &= 1.7507 \\ \log \tan 82^\circ 6' &= .8577 \\ \hline &= .6084 \\ \log \cos 12^\circ 37' &= 1.9894 \\ \hline \log x &= .6190; \\ \text{whence } x &= 4.159. \end{aligned}$$

$$19. \quad \begin{aligned} x &= \cos 28^\circ 14' \times \cos 37^\circ 26'. \\ \log \cos 28^\circ 14' &= 1.9450 \\ \log \cos 37^\circ 26' &= 1.8998 \\ \hline &= 1.8448; \\ \text{whence } x &= .6995. \end{aligned}$$

20.  $7 \log \tan x = \log 11 - \log 13.$

$$\log 11 = 1.0414$$

$$\log 13 = 1.1139$$

$$7) \overline{1.9275}$$

$$\log \tan x = \overline{1.9896};$$

$$\text{whence } x = 44^\circ 19'.$$

21.

$$\log 32.73 = 1.5149$$

$$\log 27.86 = 1.4449$$

$$\log \sin 30^\circ 16' = \overline{1.7025}$$

$$\log ab \sin C = \overline{2.6623};$$

$$\text{whence } ab \sin C = 459.5.$$

22. (i)  $na^2 \cot \frac{\pi}{n} = 32 \cot 22\frac{1}{2}^\circ$

$$= 32 \tan 67\frac{1}{2}^\circ$$

$$= 32 \times 19.3136$$

$$= 77.2544.$$

(ii)  $\frac{nr^2}{2} \sin \frac{2\pi}{n} = 5 \times (3.3)^2 \sin 36^\circ.$

$$\log 5 = .6990$$

$$2 \log 3.3 = 1.0370$$

$$\log \sin 36^\circ = \overline{1.7692}$$

$$\overline{1.5052};$$

$$\therefore \text{required value} = 32.00.$$

23.

$$\tan \phi = \frac{.7}{1 - (.35)^2} \sin 56^\circ 14'$$

$$= \frac{.7}{1.35 \times .65} \sin 56^\circ 14'.$$

$$\log .7 = \overline{1.8451}$$

$$\log \sin 56^\circ 14' = \overline{1.9198}$$

$$\overline{1.7649}$$

$$\overline{1.9432}$$

$$\overline{1.8217}; \text{ whence } \phi = 33^\circ 33'.$$

$$\log 1.35 = .1303$$

$$\log .65 = \overline{1.8129}$$

$$\overline{1.9432}$$

24.

$$l = \frac{2 \times 32.78 \times 19.23}{52.01} \cos 57^\circ 47'.$$

$$\log 32.78 = 1.5156$$

$$\log 38.46 = 1.5850$$

$$\log \cos 57^\circ 47' = \overline{1.7268}$$

$$2.8274$$

$$\log 52.01 = 1.7161$$

$$\log l = \overline{1.1113}; \text{ whence } l = 12.92.$$

25. Expression

$$= \frac{2 \times (48)^2 \sin 23^\circ}{32.19 \times (.63)^2 \cos^2 23^\circ}.$$

$$\log 2 = .3010$$

$$2 \log 48 = 3.3624$$

$$\log \sin 23^\circ = \overline{1.5919}$$

$$3.2553$$

$$1.0343$$

$$2.2210; \text{ whence value of expression} = 166.3.$$

$$\log 32.19 = 1.5077$$

$$2 \log .63 = \overline{1.5986}$$

$$2 \log \cos 23^\circ = \overline{1.9280}$$

$$\overline{1.0343}$$

**EXAMPLES. XVI. a. PAGE 166.**

$$1. \text{ First side} = \frac{s(s-a)}{c} + \frac{s(s-b)}{c} = \frac{s(2s-a-b)}{c} = s.$$

$$2. \text{ First side} = s \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \cdot \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ = s \sqrt{\frac{(s-a)^2}{s^2}} = s-a.$$

$$3. \text{ First side} = \frac{1-\cos A}{1+\cos B} = \frac{2\sin^2 \frac{A}{2}}{2\sin^2 \frac{B}{2}} = \frac{(s-b)(s-c)}{bc} \times \frac{ca}{(s-c)(s-a)} \\ = \frac{a(s-b)}{c(s-a)} = \frac{a(a+c-b)}{b(b+c-a)}.$$

$$4. \text{ First side} = \frac{b(s-b)(s-c)}{bc} + \frac{a(s-c)(s-a)}{ca} \\ = \frac{(s-c)\{s-b+s-a\}}{c} = \frac{c(s-c)}{c} = s-c.$$

$$5. \text{ Each of the expressions reduces to } \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$6. \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3 \times 14}{24 \times 7}} = \frac{1}{2}.$$

$$7. \cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{21 \times 6}{8 \times 7}} = \frac{3}{2}.$$

$$8. \text{ First side} = \frac{s(s-a) + s(s-b) + s(s-c)}{abc} \\ = \frac{s\{3s-(a+b+c)\}}{abc} = \frac{s^2}{abc}.$$

$$9. \text{ First side} = \frac{b-c}{a} \cdot \frac{s(s-a)}{bc} + \text{two similar terms} \\ = \frac{(b-c)(s^2-as)}{abc} + \text{two similar terms} \\ = \frac{s^2\{(b-c)+(c-a)+(a-b)\} - s\{a(b-c)+b(c-a)+c(a-b)\}}{abc} \\ = 0.$$

## EXAMPLES. XVI. b. PAGE 169.

$$1. \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{7 \times 4}{5 \times 8}} = \sqrt{\frac{7}{10}}.$$

$$\log \sin \frac{C}{2} = \frac{1}{2} (\log 7 - 1) \\ = 1.9225490$$

$$\log \sin 56^\circ 47' = 1.9225205 \\ \text{diff.} \quad 285$$

$$\text{prop}^l. \text{ increase} = \frac{285}{827} \times 60'' = 20.6'';$$

$$\therefore \frac{C}{2} = 56^\circ 47' 20.6'', \text{ and } C = 113^\circ 34' 41''.$$

$$\begin{array}{r} 285 \\ 60 \\ 827 \overline{) 17100} \quad (20.6 \\ 1654 \\ \hline 5600 \\ 4962 \end{array}$$

$$2. \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{16 \times 24}{67 \times 27}} = \sqrt{\frac{128}{603}}.$$

$$\log \tan \frac{A}{2} = 1.6634464 \\ = \log \tan 24^\circ 44' 13'';$$

$$\therefore A = 49^\circ 28' 26''.$$

$$\begin{array}{r} \log 128 = 2.1072100 \\ \log 603 = 2.7803173 \\ 2) 1.3268927 \\ \hline 1.6634464 \end{array}$$

$$3. \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{15 \times 5}{2 \times 2 \times 4 \times 6}} = \frac{5}{4\sqrt{2}}.$$

$$\log \cos \frac{B}{2} = \log 5 - \frac{5}{2} \log 2 \\ = 1.9463950$$

$$\log \cos 27^\circ 53' = 1.9461040 \\ \text{diff.} \quad 90$$

$$\text{prop}^l. \text{ increase} = \frac{90}{669} \times 60'' = 8.07'';$$

$$\therefore \frac{B}{2} = 27^\circ 53' 8.07'', \text{ and } B = 55^\circ 46' 16''.$$

$$\begin{array}{r} \log 5 = 0.6989700 \\ \frac{5}{2} \log 2 = 0.7525750 \\ \hline 1.9463950 \end{array}$$

$$\begin{array}{r} 30 \\ 60 \\ 223 \overline{) 1800} \quad (8.07 \\ 1784 \\ \hline 1600 \end{array}$$

$$4. \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{9 \times 2}{5 \times 6}} = \sqrt{\frac{6}{10}}.$$

$$\log \cos \frac{C}{2} = \frac{1}{2} (\log 6 - 1)$$

$$= \bar{1}.8890757$$

$$\log \cos 39^\circ 14' = \bar{1}.8890644$$

$$\text{diff.} \quad \quad \quad 113$$

$$\text{prop}^l. \text{ decrease} = \frac{113}{1032} \times 60'' = 6.6'';$$

$$\therefore \frac{C}{2} = 39^\circ 13' 53.4'', \text{ and } C = 78^\circ 27' 47''.$$

$$\begin{array}{r} 113 \\ 60 \\ 1032 \overline{) 67800} \quad (6.6 \\ \underline{6192} \\ 5880 \\ \underline{6192} \end{array}$$

$$5. \quad \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{.75 \times 2}{3.75}} = \sqrt{\frac{2}{10}}.$$

$$\log \tan \frac{C}{2} = \frac{1}{2} \{2 \log 2 - 1\} = \frac{1}{2} (\bar{1}.6020600)$$

$$= \bar{1}.8010300$$

$$\log \tan 32^\circ 18' = \bar{1}.8008365$$

$$\text{diff.} \quad \quad \quad 1935$$

$$\text{prop}^l. \text{ increase} = \frac{1935}{2796} \times 60'' = 41.5'';$$

$$\therefore \frac{C}{2} = 32^\circ 18' 41.5'', \text{ and } C = 64^\circ 37' 23''.$$

$$\begin{array}{r} 1935 \\ 60 \\ 2796 \overline{) 116100} \quad (41.5 \\ \underline{11184} \\ 4260 \\ \underline{2796} \\ 14640 \end{array}$$

$$6. \quad \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{3 \times 4}{15 \times 8}} = \sqrt{\frac{1}{10}}.$$

$$\log \tan \frac{C}{2} = -\frac{1}{2} = \bar{1}.5000000$$

$$\log \tan 17^\circ 33' = \bar{1}.500042$$

$$\text{diff.} \quad \quad \quad 42$$

$$\text{prop}^l. \text{ decrease} = \frac{42}{439} \times 60'' = 5.7'';$$

$$\therefore \frac{C}{2} = 17^\circ 32' 54.3'', \text{ and } C = 35^\circ 5' 49''.$$

$$\begin{array}{r} 42 \\ 60 \\ 439 \overline{) 2520} \quad (5.7 \\ \underline{2195} \\ 3250 \end{array}$$

7. Let  $a=4$ ,  $b=10$ ,  $c=11$ .

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{25}{2} \cdot \frac{2}{2} \cdot \frac{1}{40}} = \sqrt{\frac{15}{2^5}} = \sqrt{\frac{30}{2^6}}.$$

$$\log \cos \frac{C}{2} = \frac{1}{2} (\log 3 + 1 - 6 \log 2)$$

$$= \bar{1}.8354707$$

$$\log \cos 46^\circ 47' = \bar{1}.8355378$$

$$\text{diff.} \quad \underline{671}$$

$$\text{prop}^l. \text{ increase} = \frac{671}{1345} \times 60'' = 30''.$$

$$\therefore \frac{C}{2} = 46^\circ 47' 30'', \text{ and } C = 93^\circ 35'.$$

$$\begin{array}{r} 1 + \log 3 = 1.4771213 \\ 6 \log 2 = 1.8061800 \\ 2) \bar{1}.6709413 \\ \hline \bar{1}.8354707 \end{array}$$

$$8. \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{63 \times 7}{21 \times 8}} = \frac{1}{2}.$$

$$\log \tan \frac{B}{2} = -\log 2 = \bar{1}.6989700$$

$$\log \tan 26^\circ 33' = \bar{1}.6986847$$

$$\text{diff.} \quad \underline{2853}$$

$$\text{prop}^l. \text{ increase} = \frac{2853}{3159} \text{ of } 60'' = 54.2''.$$

$$\therefore \frac{B}{2} = 26^\circ 33' 54.2'', \text{ and } B = 53^\circ 7' 48''.$$

$$\begin{array}{r} 2853 \\ 60 \\ 3159) 171180 ( 54.2 \\ \underline{15795} \\ 13280 \\ \underline{12636} \\ 644 \end{array}$$

Again  $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{6 \times 8}{21 \times 7}} = \frac{4}{7}.$

$$\log \tan \frac{C}{2} = 2 \log 2 - \log 7$$

$$= \bar{1}.7569620$$

$$\log \tan 29^\circ 44' = \bar{1}.7567587$$

$$\text{diff.} \quad \underline{2033}$$

$$\text{prop}^l. \text{ increase} = \frac{2033}{2933} \times 60'' = 41.5''.$$

$$\therefore \frac{C}{2} = 29^\circ 44' 41.5'', \text{ and } C = 59^\circ 29' 23''.$$

$$\therefore A = 67^\circ 22' 49''.$$

$$\begin{array}{r} 2 \log 2 = .6020600 \\ \log 7 = .8450980 \\ \hline \bar{1}.7569620 \end{array}$$

$$\begin{array}{r} 2033 \\ 60 \\ 2933) 121980 ( 41.5 \\ \underline{11732} \\ 4660 \\ \underline{2933} \\ 17270 \end{array}$$

$$9. \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{3}{2} \times \frac{5}{2} \times \frac{2}{9} \times 2} = \sqrt{\frac{5}{3}}.$$

$$\log \tan \frac{B}{2} = \frac{1}{2} (\log 5 - \log 3)$$

$$= .1109244$$

$$\log \tan 52^\circ 14' = .1108395$$

$$\text{diff.} \quad 849$$

$$\log 5 = .6989700$$

$$\log 3 = .4771213$$

$$2 \quad \underline{.2218487}$$

$$.1109244$$

$$\text{prop}^l. \text{ increase} = \frac{849}{435} \times 10'' = 19.5'';$$

$$\therefore \frac{B}{2} = 52^\circ 14' 19.5'', \text{ and } B = 104^\circ 28' 39''.$$

$$\text{Again} \quad \tan \frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\frac{1}{2} \times \frac{3}{2} \times \frac{2}{9} \times \frac{2}{5}} = \frac{1}{\sqrt{3 \times 5}}.$$

$$\log \tan \frac{C}{2} = -\frac{1}{2} (\log 3 + \log 5)$$

$$= \bar{1}.4119544$$

$$\log \tan 14^\circ 28' = \bar{1}.4116146$$

$$\text{diff.} \quad 3398$$

$$\text{prop}^l. \text{ increase} = \frac{3398}{870} \times 10'' = 39';$$

$$\therefore \frac{C}{2} = 14^\circ 28' 39'', \text{ and } C = 28^\circ 57' 18''.$$

$$\therefore A = 46^\circ 34' 3''.$$

### EXAMPLES. XVI. c. PAGE 173.

$$1. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{5} \cot 30^\circ = \frac{2}{10} \sqrt{3}.$$

$$\log \tan \frac{A-B}{2} = \log 2 + \frac{1}{2} \log 3 - 1$$

$$= \bar{1}.5395907$$

$$\log \tan 19^\circ 6' = \bar{1}.5394287$$

$$\text{diff.} \quad 1620$$

$$\text{prop}^l. \text{ increase} = \frac{1620}{4081} \times 60'' = 24''.$$

$$\log 2 = .3010300$$

$$\frac{1}{2} \log 3 = .2385607$$

$$\underline{.5395907}$$

$$\therefore \frac{A-B}{2} = 19^{\circ} 6' 24'', \text{ and } \frac{A+B}{2} = 60^{\circ};$$

$$\therefore A = 79^{\circ} 6' 24''; B = 40^{\circ} 53' 36''.$$

2.

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{8}{10} \cot 32^{\circ} 30'.$$

$$\log \tan \frac{C-A}{2} = 3 \log 2 - 1 + \log \cot 32^{\circ} 30'$$

$$\begin{array}{r} = .0989027 \\ \log \tan 51^{\circ} 28' = .0988763 \\ \text{diff.} \quad \quad \quad 264 \end{array}$$

$$\text{prop}^l. \text{ increase} = \frac{264}{2592} \times 60'' = 6'';$$

$$\begin{array}{r} 3 \log 2 = .9030900 \\ \log \cot 32^{\circ} 30' = .1958127 \\ \hline 1.0989027 \end{array}$$

$$\therefore \frac{C-A}{2} = 51^{\circ} 28' 6'',$$

$$\frac{C+A}{2} = 57^{\circ} 33';$$

$$\therefore C = 108^{\circ} 58' 6''; A = 6^{\circ} 1' 54''.$$

3.

$$\tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{5}{12} \sqrt{3}.$$

$$\begin{array}{r} \log \tan \frac{B-A}{2} = 1 - 3 \log 2 - \frac{1}{2} \log 3 \\ = 1.8583491 \end{array}$$

$$\begin{array}{r} \log \tan 35^{\circ} 49' = 1.8583357 \\ \text{diff.} \quad \quad \quad 137 \end{array}$$

$$\text{prop}^l. \text{ increase} = \frac{137}{2662} \times 60'' = 3'';$$

$$\therefore \frac{B-A}{2} = 35^{\circ} 49' 3'',$$

$$\frac{B+A}{2} = 60^{\circ};$$

$$\therefore B = 95^{\circ} 49' 3''; A = 24^{\circ} 10' 57''$$

$$\begin{array}{r} \frac{1}{2} \log 3 = .2385606 \\ 3 \log 2 = .9030900 \\ \hline 1.1416506 \end{array}$$

$$4. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{4}{50} \cot 22^\circ 15' = \frac{8}{100} \cot 22^\circ 15'.$$

$$\log \tan \frac{B-C}{2} = \bar{1}.2912491$$

$$\log \tan 11^\circ 3' = \bar{1}.2906713$$

$$\text{diff.} \quad \frac{5778}{6711}$$

$$\text{prop}^l. \text{ increase} = \frac{5778}{6711} \times 60'' = 52'';$$

$$\begin{array}{r} 3 \log 2 - 2 = \bar{2}.9030900 \\ \log \cot 22^\circ 15' = \cdot 3881591 \\ \hline \bar{1}.2912491 \end{array}$$

$$\therefore \frac{B-C}{2} = 11^\circ 3' 52'', \text{ and } \frac{B+C}{2} = 67^\circ 45';$$

$$\therefore B = 78^\circ 48' 52''; C = 56^\circ 41' 8''.$$

$$5. \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{10}{32} \cot 17^\circ 21' 15''.$$

$$\begin{aligned} \log \tan \frac{C-A}{2} &= 1 + \log \cot 17^\circ 21' 15'' - 5 \log 2 \\ &= 1.5051500 - 1.5051500 \\ &= 0. \end{aligned}$$

$$\therefore \frac{C-A}{2} = 45^\circ, \text{ and } \frac{C+A}{2} = 72^\circ 38' 45'';$$

$$\therefore C = 117^\circ 38' 45''; A = 27^\circ 38' 45''.$$

$$6. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{4} \cot 30^\circ 15'.$$

$$\begin{aligned} \log \tan \frac{A-B}{2} &= -2 \log 2 + \log \cot 30^\circ 15' \\ &= \bar{1}.63214 \end{aligned}$$

$$\begin{array}{r} \log \tan 23^\circ 13' = \bar{1}.63240 \\ \text{diff.} \quad \frac{26}{35} \end{array}$$

$$\begin{array}{r} \log \cot 30^\circ 15' = \cdot 23420 \\ 2 \log 2 = \cdot 60206 \\ \hline 1.63214 \end{array}$$

$$\text{prop}^l. \text{ decrease} = \frac{26}{35} \times 60'' = 45'';$$

$$\therefore \frac{A-B}{2} = 23^\circ 12' 15'', \text{ and } \frac{A+B}{2} = 59^\circ 45';$$

$$\therefore A = 82^\circ 57' 15''; B = 36^\circ 32' 45''.$$

$$7. \quad \tan \frac{A-C}{2} = \frac{a-c}{a+c} \cot \frac{B}{2} = \frac{71}{283} \cot 28^\circ 14'.$$

$$\log \tan \frac{A-C}{2} = 1.3556602$$

$$\log \tan 12^\circ 46' = 1.3552267$$

$$\text{diff.} \quad 4335$$

$$\text{prop}^l. \text{ increase} = \frac{4335}{5859} \times 60'' = 44'';$$

$$\therefore \frac{A-C}{2} = 12^\circ 46' 44'', \text{ and } \frac{A+C}{2} = 61^\circ 46';$$

$$\therefore A = 74^\circ 32' 44''; C = 48^\circ 59' 16''.$$

$$\log 71 = 1.8512583$$

$$\log \cot \frac{B}{2} = \frac{.2700705}{2.1213288}$$

$$\log 583 = 2.7656686$$

$$1.3556602$$

8.

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{3}{5} \cot 32^\circ 30'.$$

$$\log \tan \frac{B-C}{2} = 1.9739640$$

$$\log \tan 43^\circ 18' = 1.9742133$$

$$\text{diff.} \quad 2493$$

$$\text{prop}^l. \text{ decrease} = \frac{2493}{2531} \times 60'' = 59'';$$

$$\therefore \frac{B-C}{2} = 43^\circ 17' 1'', \text{ and } \frac{B+C}{2} = 57^\circ 30';$$

$$\therefore B = 100^\circ 47' 1''; C = 14^\circ 12' 59''.$$

$$\log 3 = .4771213$$

$$\log 5 = .6989700$$

$$1.7781513$$

$$\log \cot 32^\circ 30' = \frac{.1958127}{1.9739640}$$

$$1.9739640$$

9. Here

$$\cot \frac{A-B}{2} = \frac{a+b}{a-b} \tan \frac{C}{2} = 2 \tan \frac{C}{2}.$$

$$\therefore \log \cot \frac{A-B}{2} = \log 2 + \log \tan 15^\circ 5' 2.5''$$

$$= 1.7316236$$

$$\log \cot 61^\circ 41' = 1.7314436$$

$$\text{diff.} \quad 1800$$

$$\text{prop}^l. \text{ decrease} = \frac{1800}{504} \times 10'' = 35.7'';$$

$$\therefore \frac{A-B}{2} = 61^\circ 40' 24.3'', \text{ and } \frac{A+B}{2} = 74^\circ 54' 51.5'';$$

$$\therefore A = 136^\circ 35' 21.8''; B = 13^\circ 14' 33.2''.$$

$$\log 2 = .3010300$$

$$\log \tan 15^\circ 5' = 1.4305727$$

$$\frac{2.5}{10} \times 838 = \frac{209}{1.7316236}$$

$$1.7316236$$

## EXAMPLES. XVI. d. PAGE 174.

1. Here  $A = 180^\circ - 114^\circ 45' = 65^\circ 15'$ .

$$c = \frac{a \sin C}{\sin A} = \frac{100 \sin 54^\circ 30'}{\sin 65^\circ 15'}.$$

$$\begin{aligned} \log c &= 1.9525317 \\ &= \log 89.646162; \\ \therefore c &= 89.646162. \end{aligned}$$

$$\begin{aligned} \log \sin 54^\circ 30' &= \bar{1}.9106860 \\ \log 100 &= 2 \\ &\underline{1.9106860} \\ \log \sin 65^\circ 15' &= \bar{1}.9581543 \\ &\underline{1.9525317} \end{aligned}$$

$$2. \quad a = \frac{c \sin A}{\sin C} = \frac{270 \sin 55^\circ}{\sin 60^\circ} = 270 \sin 55^\circ \times \frac{2}{\sqrt{3}}.$$

$$\therefore \log a = 1 + 3 \log 3 + \log \sin 55^\circ - \frac{1}{2} \log 3 + \log 2$$

$$\begin{aligned} &= 2.4071977 \\ \log 255.38 &= 2.4071869 \\ \text{diff.} &\quad \underline{108} \end{aligned}$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{108}{170} \times .01 = .0064;$$

$$\therefore a = 255.3864.$$

$$\begin{aligned} \log 270 &= 2.4313639 \\ \log \sin 55 &= \bar{1}.9133645 \\ \log 2 &= .3010300 \\ &\underline{2.6457584} \\ \frac{1}{2} \log 3 &= .2385607 \\ &\underline{2.4071977} \end{aligned}$$

$$3. \quad c = \frac{b \sin C}{\sin B} = \frac{100 \sin 62^\circ 5'}{\sin 72^\circ 14'}.$$

$$\begin{aligned} \log c &= 1.96749 \\ &= \log 92.788; \\ \therefore c &= 92.788. \end{aligned}$$

$$\begin{aligned} \log \sin 62^\circ 5' &= \bar{1}.94627 \\ \log 100 &= 2 \\ &\underline{1.94627} \\ \log \sin 72^\circ 14' &= \bar{1}.97878 \\ &\underline{1.96749} \end{aligned}$$

4. Here  $A = 180^\circ - 148^\circ 40' = 31^\circ 20'$ .

$$b = \frac{a \sin B}{\sin A} = \frac{102 \sin 70^\circ 30'}{\sin 31^\circ 20'}.$$

$$\begin{aligned} \log b &= 2.267 \\ &= \log 185; \\ \therefore b &= 185. \end{aligned}$$

$$\begin{aligned} \log 102 &= 2.009 \\ \log \sin 70^\circ 30' &= \bar{1}.974 \\ &\underline{1.983} \\ \log \sin 31^\circ 20' &= \bar{1}.716 \\ &\underline{2.267} \end{aligned}$$

$$\begin{aligned}\text{Again } c &= \frac{a \sin C}{\sin A}, \\ \log c &= 2.283 \\ &= \log 192; \\ \therefore c &= 192.\end{aligned}$$

$$\begin{aligned}\log 102 &= 2.009 \\ \log \sin 78^\circ 10' &= \bar{1}.990 \\ &\quad \underline{1.999} \\ \log \sin 31^\circ 20' &= \bar{1}.716 \\ &\quad \underline{2.283}\end{aligned}$$

5. Here  $c = \frac{a \sin C}{\sin A} = \frac{123}{\sqrt{2} \sin 15^\circ 43'}.$

$$\begin{aligned}\log c &= 2.5066124 \\ \log 321.10 &= 2.5066403 \\ \text{diff.} &\quad \underline{279} \\ \text{prop}^l. \text{ decrease} &= \frac{279}{135} \times .01 = .02066. \\ \text{Thus } c &= 321.0793.\end{aligned}$$

$$\begin{aligned}\frac{1}{2} \log 2 &= .1505150 \\ \log \sin 15^\circ 43' &= \bar{1}.4327777 \\ &\quad \underline{\bar{1}.5832927} \\ \log 123 &= 2.0899051 \\ &\quad \underline{2.5066124}\end{aligned}$$

6.  $a = \frac{b \sin A}{\sin B} = \frac{1006.62 \sin 44^\circ}{\sin 66^\circ}.$

$$\begin{aligned}\log 1006.62 &= 3.0028656 \\ \log \sin 44^\circ &= \bar{1}.8417713 \\ &\quad \underline{2.8446369} \\ \log \sin 66^\circ &= \bar{1}.9607302 \\ \log a &= 2.8839067 \\ \therefore a &= 765.4321.\end{aligned}$$

$$\begin{aligned}c &= \frac{b \sin C}{\sin B} = \frac{1006.62 \sin 70^\circ}{\sin 66^\circ}, \\ \log 1006.62 &= 3.0028656 \\ \log \sin 70^\circ &= \bar{1}.9729858 \\ &\quad \underline{2.9758514} \\ \log \sin 66^\circ &= \bar{1}.9607302 \\ \log c &= 3.0151212 \\ \therefore c &= 1035.43.\end{aligned}$$

7. Here  $A =$  supplement of  $75^\circ 45'$ ;

$$\begin{aligned}\therefore b &= \frac{1652 \sin 26^\circ 30'}{\sin 75^\circ 45'}, \\ \log b &= 2.8852436 \\ \log 767.80 &= 2.8852481 \\ \text{diff.} &\quad \underline{45} \\ \text{prop}^l. \text{ decrease} &= \frac{45}{57} \times .01 = .008; \quad \therefore b = 767.792.\end{aligned}$$

$$\begin{aligned}\log 1652 &= 3.2180100 \\ \log \sin 26^\circ 30' &= \bar{1}.6495274 \\ &\quad \underline{2.8675374} \\ \log \sin 73^\circ 45' &= \bar{1}.9822938 \\ &\quad \underline{2.8852436}\end{aligned}$$

$$\begin{aligned}\text{Again } c &= \frac{1652 \sin 47^\circ 15'}{\sin 73^\circ 45'}; \\ \log c &= 3.1016030 \\ \log 1263.6 &= 3.1016096 \\ \text{diff.} &\quad \underline{66}\end{aligned}$$

$$\begin{aligned}\log 1652 &= 3.2180100 \\ \log \sin 47^\circ 15' &= \bar{1}.8658868 \\ &\quad \underline{3.0838968} \\ \log \sin 73^\circ 45' &= \bar{1}.9822938 \\ &\quad \underline{3.1016030}\end{aligned}$$

$$\text{prop}^l. \text{ decrease} = \frac{66}{344} \times .1 = .019; \quad \therefore c = 1263.58.$$

## EXAMPLES. XVI. e. PAGE 176.

$$1. \quad \sin A = \frac{a \sin B}{b} = \frac{145}{178} \sin 41^\circ 10'.$$

$$\log \sin A = \bar{1}.7293399,$$

$$\therefore A = 32^\circ 25' 35''.$$

$$\log 145 = 2.1613680$$

$$\log \sin 41^\circ 10' = \bar{1}.8183919$$

$$\underline{1.9797599}$$

$$\log 178 = 2.2504200$$

$$\underline{\bar{1}.7293399}$$

$$2. \quad \sin B = \frac{b}{a} \sin A = \frac{127}{85} \sin 26^\circ 26'.$$

$$\log \sin B = \bar{1}.8228972,$$

$$\therefore B = 41^\circ 41' 28'';$$

and since  $a$  is  $< b$ , there is another value of  $B$ , namely,

$$B = 138^\circ 18' 32''.$$

$$\log 127 = 2.1038037$$

$$\log \sin 26^\circ 26' = \bar{1}.6485124$$

$$\underline{1.7523161}$$

$$\log 85 = 1.9294189$$

$$\underline{\bar{1}.8228972}$$

$$3. \quad \sin B = \frac{b}{c} \sin C = \frac{4}{5} \sin 45^\circ = \frac{8}{10} \cdot \frac{1}{\sqrt{2}},$$

$$\log \sin B = 3 \log 2 - 1 - \frac{1}{2} \log 2.$$

$$\therefore \log \sin B = \bar{1}.7525750.$$

$$\therefore B = 34^\circ 26'$$

and

$$A = 100^\circ 34'.$$

$$3 \log 2 = .9030900$$

$$1 + \frac{1}{2} \log 2 = \frac{1.1505150}{\underline{\bar{1}.7525750}}$$

$$4. \quad \sin B = \frac{b}{a} \sin A = \frac{1706}{1405} \sin 40^\circ.$$

$$\log \sin B = \bar{1}.8923702$$

$$\log \sin 51^\circ 18' = \bar{1}.8923342$$

$$\text{diff.} \quad \underline{360}$$

$$\text{prop}^l. \text{ increase} = \frac{360}{1012} \times 60'' = 21''.$$

$$\log 1706 = 3.2319790$$

$$\log \sin 40^\circ = \bar{1}.8080675$$

$$\underline{3.0400465}$$

$$\log 1405 = 3.1476763$$

$$\underline{\bar{1}.8923702}$$

$\therefore B = 51^\circ 18' 21''$ ; but since  $a < b$ , there is another value of  $B$ , namely,  $128^\circ 41' 39''$ .

$$5. \quad \sin C = \frac{c}{b} \sin B = \frac{394}{573} \sin 112^\circ 4' = \frac{394}{573} \cos 22^\circ 4'.$$

$$\therefore \log \sin C = \bar{1}.8043030$$

$$\log \sin 39^\circ 35' = \bar{1}.8042757$$

$$\text{diff.} \quad \underline{\quad 273 \quad}$$

$$\text{prop}^l. \text{ increase} = \frac{273}{1527} \times 60'' = 11'';$$

$$\therefore C = 39^\circ 35' 11''; \text{ and } A = 28^\circ 20' 49''.$$

$$\log 394 = 2.5954962$$

$$\log \sin 112^\circ 4' = \bar{1}.9669614$$

$$\underline{\quad 2.5624576 \quad}$$

$$\log 573 = 2.7581546$$

$$\underline{\quad \bar{1}.8043030 \quad}$$

$$6. \quad \sin C = \frac{c}{b} \sin B = \frac{12}{8.4} \sin 37^\circ 36' = \frac{1}{.7} \sin 37^\circ 36'.$$

$$\log \sin C = \bar{1}.9403352$$

$$\log \sin 60^\circ 39' = \bar{1}.9403381$$

$$\text{diff.} \quad \underline{\quad 29 \quad}$$

$$\text{prop}^l. \text{ decrease} = \frac{29}{711} \times 60'' = 2.4''.$$

$$\log \sin 37^\circ 36' = \bar{1}.7854332$$

$$\log .7 = \bar{1}.8450980$$

$$\underline{\quad \bar{1}.9403352 \quad}$$

$\therefore C = 60^\circ 38' 58''$ ; but since  $b < c$ , there is another value of  $C$ , namely,  $119^\circ 21' 2''$ . Thus  $A = 81^\circ 45' 2''$ , or  $23^\circ 2' 58''$ .

$$7. \quad (i) \quad \text{Here } \sin C = \frac{c \sin A}{a} = \frac{250}{125} \times \frac{1}{2} = 1.$$

$$\therefore C = 90^\circ, \text{ and there is no ambiguity.}$$

(ii) Here  $\sin C = \frac{250}{200} \times \frac{1}{2} = \frac{5}{8}$ , and since  $a < c$  there will be two values of  $C$  satisfying the data.

(iii) Here  $\sin C = \frac{125}{200} \times \frac{1}{2} = \frac{5}{16}$ ; but since  $a > c$  there is only one solution.

$$\text{From (ii) we have } \log \sin C = \log 5 - \log 8 = \bar{1}.79588.$$

$$\therefore C = 38^\circ 41', \text{ or } 141^\circ 19'; \text{ and } A = 111^\circ 19', \text{ or } 8^\circ 41'.$$

Now in the obtuse-angled triangle we have

$$b = \frac{a \sin B}{\sin A} = \frac{200 \sin 8^\circ 41'}{\sin 30^\circ}.$$

$$\log b = 1.7809601$$

$$\log 60.389 = 1.7809578$$

$$\text{diff.} \quad \underline{\quad 23 \quad}$$

$$\text{prop}^l. \text{ increase} = \frac{23}{72} \times .001 = .0003.$$

$$\therefore b = 60.3893.$$

$$\log 200 = 2.3010300$$

$$\log \sin 8^\circ 41' = \bar{1}.1789001$$

$$\underline{\quad 1.4799301 \quad}$$

$$\log \sin 30^\circ = \bar{1}.6989700$$

$$\underline{\quad 1.7809601 \quad}$$

**EXAMPLES. XVI. f. PAGE 180.**

1.

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3 \cdot 4}{12 \cdot 5}} = \sqrt{\frac{1}{5}};$$

$$\therefore \log \tan \frac{B}{2} = -\frac{1}{2} \log 5 = -\frac{1}{2}(1 - \log 2)$$

$$= \bar{1}.6505150$$

$$\log \tan 24^\circ 5' = \bar{1}.6502809$$

$$\text{diff.} \quad \quad \quad 2341$$

$$\text{prop}^l. \text{ increase} = \frac{2341}{3390} \times 60'' = 41.4'';$$

$$\therefore \frac{B}{2} = 24^\circ 5' 41.4'', \text{ and } B = 48^\circ 11' 23''.$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{4 \cdot 5}{12 \cdot 3}} = \sqrt{\frac{5}{3}};$$

$$\therefore \log \tan \frac{C}{2} = \frac{1}{2}(1 - \log 2 - 2 \log 3)$$

$$= \bar{1}.8723637$$

$$\log \tan 36^\circ 41' = \bar{1}.8721123$$

$$\text{diff.} \quad \quad \quad 2514$$

$$\text{prop}^l. \text{ increase} = \frac{2514}{2637} \times 60'' = 57.2'';$$

$$\therefore \frac{C}{2} = 36^\circ 41' 57.2'', \text{ and } C = 73^\circ 23' 54'';$$

$$\therefore A = 58^\circ 24' 43''.$$

2.

$$\cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2}$$

$$= \frac{512}{162} \tan 12^\circ = \frac{256}{81} \tan 12^\circ.$$

$$\therefore \log \cot \frac{A}{2} = 8 \log 2 - 4 \log 3 + \log \tan 12^\circ$$

$$= \bar{1}.8272293 = \log \cot 56^\circ 6' 27'';$$

$$\therefore \frac{A}{2} = 56^\circ 6' 27'', \text{ and } A = 112^\circ 12' 54''.$$

$$\therefore B + C = 67^\circ 47' 6'', \text{ and } B - C = 24^\circ;$$

$$\therefore B = 45^\circ 53' 33'', \text{ and } C = 21^\circ 53' 33''.$$

$$3. \quad \sin A = \frac{2}{7}, \text{ if } A \text{ is the less of the two acute angles.}$$

$$\log \sin A = \log 2 - \log 7$$

$$= \bar{1}.455932$$

$$\log \sin 14^\circ 11' = \bar{1}.455921$$

$$\text{diff.} \quad \quad \quad 11$$

$$\text{prop}^t. \text{ increase} = \frac{11}{110} \times 60'' = 6''.$$

$$\therefore A = 14^\circ 11' 6'';$$

$$\therefore B = 90^\circ - 14^\circ 11' 6'' = 75^\circ 48' 54''.$$

$$4. \quad \text{Here } a = 2183, A = 30^\circ 22', B = 78^\circ 14', C = 71^\circ 24'.$$

$$b = \frac{a \sin B}{\sin A} = \frac{2183 \sin 78^\circ 14'}{\sin 30^\circ 22'}.$$

$$\log b = 3.6260817$$

$$\log 4227.4 = 3.6260733$$

$$\text{diff.} \quad \quad \quad 84$$

$$\text{prop}^t. \text{ increase} = \frac{84}{103} \times .1 = .0815;$$

$$\therefore b = 4227.4815.$$

$$\log 2183 = 3.3390537$$

$$\log \sin B = \bar{1}.9907766$$

$$3.3298303$$

$$\log \sin A = \bar{1}.7037486$$

$$3.6260817$$

$$5. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{9} \cot 11^\circ 10'.$$

$$\therefore \log \tan \frac{B-C}{2} = \log \cot 11^\circ 10' - 2 \log 3$$

$$= .70465 - .95424 = \bar{1}.75041.$$

$$\therefore \frac{B-C}{2} = 29^\circ 22' 26'', \text{ and } \frac{B+C}{2} = 78^\circ 50';$$

$$\therefore B = 108^\circ 12' 26''; C = 49^\circ 27' 34''.$$

$$\text{Now} \quad a = \frac{c \sin A}{\sin C};$$

$$\therefore \log a = \log 2 + \log \sin 22^\circ 20' - \log \sin 49^\circ 27' 34''$$

$$= .30103 + \bar{1}.57977 - \bar{1}.88079$$

$$= .00001;$$

$$\therefore a = 1, \text{ approximately.}$$

$$6. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = .56234 \cot 29^\circ 21' 8''.$$

Now

$$\log \cot 29^\circ 21' = .250015$$

$$\log \cot 29^\circ 22'' = .249715$$

$$\text{diff. for } 60'' \quad \underline{300}$$

$$\therefore \text{prop}^l. \text{ decrease for } 3'' = \frac{3}{60} \times 300 = 15;$$

$$\therefore \log \cot 29^\circ 21' 3'' = .250000.$$

$$\therefore \log \tan \frac{A-B}{2} = \bar{1}.75 + .25 = 0.$$

$$\therefore \tan \frac{A-B}{2} = 1, \text{ so that } \frac{A-B}{2} = 45^\circ.$$

$$\text{Also } \frac{A+B}{2} = 60^\circ 38' 57''; \text{ whence } A = 105^\circ 38' 57'', B = 15^\circ 38' 57''.$$

$$7. \quad \sin B = \frac{b \sin A}{a} = \frac{12 \sin 30^\circ}{9};$$

$$\therefore \log \sin B = 1.07918 + \bar{1}.69897 - .95424 = \bar{1}.82391;$$

$$\therefore B = 41^\circ 48' 39'' \text{ or } 138^\circ 11' 21'', \text{ both values being admissible since } a < b.$$

$$\therefore C = 108^\circ 11' 21'' \text{ or } 11^\circ 48' 39''.$$

$$\text{Again} \quad c = \frac{b \sin C_1}{\sin B_1} = \frac{12 \sin 108^\circ 11' 21''}{\sin 41^\circ 48' 39''};$$

$$\therefore \log c = 1.07918 + \bar{1}.97774 - \bar{1}.82391 = 1.23301;$$

$$\therefore c = 17.1.$$

$$\text{Similarly from } c = \frac{b \sin C_2}{\sin B_2}, \text{ we easily obtain } c = 3.68.$$

$$8. \quad \tan \frac{C}{2} = \frac{a-b}{a+b} \cot \frac{A-B}{2} = \frac{1}{2} \cot 45^\circ = \frac{1}{2};$$

$$\therefore \log \tan \frac{C}{2} = \log 1 - \log 2 = \bar{1}.6989700$$

$$\begin{array}{r} \log \tan 26^\circ 33' \\ \text{diff.} \end{array} \quad \begin{array}{r} = \bar{1}.6986847 \\ \underline{2853} \end{array}$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2853}{3159} \times 60'' = 54.2'';$$

$$\therefore \frac{C}{2} = 26^\circ 33' 54.2'', \text{ and } C = 53^\circ 7' 48''.$$

Hence  $\frac{A+B}{2} = 63^{\circ} 26' 6''$ , and  $\frac{A-B}{2} = 45^{\circ}$ ;

$$\therefore A = 108^{\circ} 26' 6'', B = 18^{\circ} 26' 6''.$$

9. (1) Let  $a = 1404$ ,  $b = 960$ ,  $A = 32^{\circ} 15'$ ;

then  $\sin B = \frac{b \sin A}{a} = \frac{80}{117 \operatorname{cosec} 32^{\circ} 15'}$ ;

$$\therefore \log \sin B = 1 + 3 \log 2 - (2 \log 3 + \log 13 + \log \operatorname{cosec} 32^{\circ} 15') \\ = \bar{1}.5621316, \text{ on reduction.}$$

$$\therefore B = 21^{\circ} 23'; \quad \therefore C = 126^{\circ} 22'.$$

(2) Let  $a = 1404$ ,  $b = 960$ ,  $B = 32^{\circ} 15'$ ;

then  $\sin A = \frac{117}{80 \operatorname{cosec} 32^{\circ} 15'}$ ;

$$\therefore \log \sin A = 2 \log 3 + \log 13 - (1 + 3 \log 2 + \log \operatorname{cosec} 32^{\circ} 15') \\ = \bar{1}.8923236, \text{ on reduction.}$$

$$\therefore A = 51^{\circ} 18', \text{ or } 128^{\circ} 42' \text{ since the solution is ambiguous.}$$

$$\therefore C = 96^{\circ} 27', \text{ or } 19^{\circ} 3'.$$

10. We have  $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1-\frac{c}{b}}{1+\frac{c}{b}} \cot \frac{A}{2}$

$$= \frac{1-\cos \phi}{1+\cos \phi} \cot \frac{A}{2} = \tan^2 \frac{\phi}{2} \cot \frac{A}{2},$$

where  $\cos \phi = \frac{c}{b} = \frac{10}{11}$ .

Hence  $\log \cos \phi = 1 - \log 11 = \bar{1}.958607$ ;

$$\therefore \phi = 24^{\circ} 37' 12''.$$

Again  $\log \tan \frac{B-C}{2} = 2 \log \tan \frac{\phi}{2} + \log \cot \frac{A}{2}$

$$= \bar{2}.677782 + .495800$$

$$= \bar{1}.173582.$$

$$\therefore \frac{B-C}{2} = 8^{\circ} 28' 56.5'', \text{ and } \frac{B+C}{2} = 72^{\circ} 17' 30'';$$

$$\therefore B = 80^{\circ} 46' 26.5'', C = 63^{\circ} 48' 33.5''.$$

$$11. \quad \sin B = \frac{b}{a} \sin A = \frac{1071}{873} \sin 50^\circ;$$

$$\therefore \log \sin B = 3.029789 + \bar{1}.884254 - 2.941014$$

$$= \bar{1}.973029$$

$$\log \sin 70^\circ = \bar{1}.972986$$

$$\text{diff.} \quad \quad \quad 43$$

$$\log \sin 70^\circ 1' = \bar{1}.973032$$

$$\log \sin 70^\circ = \bar{1}.972986$$

$$\text{diff. for } 1' \quad \quad \quad 46$$

$$\text{prop}^l. \text{ increase} = \frac{43}{46} \times 60'' = 56'';$$

$\therefore B = 70^\circ 0' 56''$ , or  $109^\circ 59' 4''$ , both values being admissible since  $a < b$ .

12. As in Art. 195 we easily obtain

$$c^2 = (a+b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left( \frac{a-b}{a+b} \right)^2 \cot^2 \frac{C}{2} \right\}.$$

Now let

$$\frac{a-b}{a+b} \cot \frac{C}{2} = \tan \theta \dots\dots\dots(1),$$

then

$$c = (a+b) \sin \frac{C}{2} \sec \theta \dots\dots\dots(2).$$

From (1),

$$\tan \theta = \frac{1}{3} \cot 18^\circ 26' 6'',$$

$$\therefore \log \tan \theta = .47712 - .47712 = 0.$$

Whence

$$\tan \theta = 1, \text{ and } \theta = 45^\circ.$$

From (2),

$$c = 9 \sqrt{2} \sin 18^\circ 26' 6'';$$

$$\therefore \log c = 2 \log 3 + \frac{1}{2} \log 2 + \log \sin 18^\circ 26' 6''$$

$$= .95424 + .15052 + \bar{1}.5$$

$$= .60476.$$

$$\therefore c = 4.0249.$$

$$13. \text{ Here } \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \text{ and } s-a=549, s-c=291;$$

$$\therefore \log \sin \frac{B}{2} = \frac{1}{2} (\log 549 + \log 291 - \log 1000 - \log 1258).$$

$$\log 549 = 2.7395723$$

$$\log 291 = 2.4638930$$

$$5.2034653$$

$$6.0996806$$

$$2) \bar{1}.1037847$$

$$\log \sin \frac{B}{2} = \bar{1}.5518924$$

$$\log \sin 20^\circ 52' = \bar{1}.5516871$$

$$\text{diff.} \quad \quad \quad 2053$$

$$\log 1000 = 3$$

$$\log 1258 = 3.0996806$$

$$6.0996806$$

$$\text{Diff. for } 60'' = 3313;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2053}{3313} \times 60'' = 37''.$$

$$\therefore \frac{B}{2} = 20^\circ 52' 37'', \text{ and } B = 41^\circ 45' 14''.$$

$$14. \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{326 \times 199}{976 \times 451}}.$$

$$\log 326 = 2.5132176$$

$$\log 199 = 2.2988531$$

$$4.8120707$$

$$5.6436263$$

$$2) \bar{1}.1684441$$

$$\log \tan \frac{A}{2} = \bar{1}.5842222$$

$$\log \tan 21^{\circ} 0' = \bar{1}.5841774$$

$$\text{diff.} \quad 448$$

$$\log 976 = 2.9894498$$

$$\log 451 = 2.6541765$$

$$5.6436263$$

$$\text{Diff. for } 20'' = 3775;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{448}{3775} \times 60'' = 7'';$$

$$\therefore \frac{A}{2} = 21^{\circ} 0' 7'', \text{ and } A = 42^{\circ} 0' 14''.$$

$$\text{Again } \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{451 \times 199}{976 \times 326}}.$$

$$\log 199 = 2.2988531$$

$$\log 451 = 2.6541765$$

$$4.9530296$$

$$5.5026674$$

$$2) \bar{1}.4503622$$

$$\log \tan \frac{B}{2} = \bar{1}.7251811$$

$$\log \tan 27^{\circ} 58' = \bar{1}.7250646$$

$$\text{diff.} \quad 1165$$

$$\log 976 = 2.9894498$$

$$\log 326 = 2.5132176$$

$$5.5026674$$

$$\text{Diff. for } 60'' = 3.049;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{1165}{3049} \times 60'' = 23'';$$

$$\therefore \frac{B}{2} = 27^{\circ} 58' 23'', \text{ and } B = 55^{\circ} 56' 46''.$$

$$\therefore C = 82^{\circ} 3'.$$

$$15. \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{2\frac{1}{2} \times 1\frac{1}{2}}{5 \times 6}} = \sqrt{\frac{1}{8}};$$

$$\therefore \log \sin \frac{A}{2} = -\frac{3}{2} \log 2$$

$$= \bar{1}.5484550,$$

$$\log \sin 20^{\circ} 42' = \bar{1}.5483585$$

$$\text{diff.} \quad 965$$

$$\text{Diff. for } 1' = 3342;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{965}{3342} \times 60'' = 17.3'';$$

$$\therefore A = 41^{\circ} 24' 35''.$$

$$16. \text{ Here } \sin C = \frac{c}{b} = \frac{28.58}{57.321}.$$

$$\begin{array}{r} \log 28.58 = 1.4560622 \\ \log 57.321 = 1.7583138 \\ \hline \log \sin 29^\circ 54' = 1.6977484 \\ \log \sin 29^\circ 54' = 1.6976545 \\ \text{diff.} \qquad \qquad \qquad 939 \end{array}$$

$$\begin{array}{l} \text{Diff. for } 60'' = 2196; \\ \therefore \text{prop}^l. \text{ increase} = \frac{939}{2196} \times 60'' = 26''. \end{array}$$

$$\therefore C = 29^\circ 54' 26''; \text{ whence } A = 60^\circ 5' 34''.$$

17. Let  $C$  be the right angle, and  $A = 18^\circ 37' 29''$ ; then

$$c = \frac{a}{\sin A} = \frac{284}{\sin 18^\circ 37' 29''}.$$

$$\begin{array}{r} \log 284 = 2.4533183 \\ \log \sin 18^\circ 37' 29'' = 1.5042917 \\ \hline \log c = 2.9490266 \\ \log 889.25 = 2.9490239 \\ \hline \qquad \qquad \qquad 27 \\ \qquad \qquad \qquad 5 \qquad \qquad 25 \\ \qquad \qquad \qquad \qquad \qquad 20 \\ \qquad \qquad \qquad 4 \qquad \qquad 20 \end{array}$$

$$\begin{array}{r} \log \sin 18^\circ 37' = 1.5041105 \\ \frac{29}{60''} \times 3748 = 1812 \\ \hline \log \sin 18^\circ 37' 29'' = 1.5042917 \end{array}$$

$$\therefore c = 889.2554 \text{ feet.}$$

$$18. \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{8} \cdot \sqrt{3};$$

$$\begin{array}{r} \therefore \log \tan \frac{B-C}{2} = \frac{1}{2} \log 3 - 3 \log 2 \\ \qquad \qquad \qquad = 1.3354706 \\ \log \tan 12^\circ 12' = 1.3348711 \\ \text{diff.} \qquad \qquad \qquad 5995 \end{array}$$

$$\begin{array}{l} \text{Diff. for } 60'' = 6112; \\ \therefore \text{prop}^l. \text{ increase} = \frac{5995}{6112} \times 60'' = 59''. \end{array}$$

$$\therefore \frac{B-C}{2} = 12^\circ 12' 59'', \text{ and } \frac{B+C}{2} = 60^\circ;$$

$$\therefore B = 72^\circ 12' 59'', C = 47^\circ 47' 1''.$$

19. Let  $AC$  be the ladder,  $C$  the window, and  $B$  the foot of the wall, then from the right-angled triangle  $ABC$ ,

$$\begin{aligned} AC = b &= \frac{42.37}{\sin 72^\circ 15'}, \\ \log b &= 1.6270585 - 1.9788175 \\ &= 1.6482410 \\ \log 44.487 &= \frac{1.6482331}{79} \\ &\quad \quad \quad \frac{8}{78} \\ \therefore \text{length of ladder} &= 44.4878 \text{ feet.} \end{aligned}$$

$$20. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{9.99}{53.91} \cot 17^\circ 30'.$$

$$\begin{aligned} \log 9.99 &= .9995655 \\ \log \cot 17^\circ 30' &= .5012777 \\ &\quad 1.5008432 \\ \log 53.91 &= 1.7316693 \\ &\quad 1.7691739 \\ \log \tan 30^\circ 26' &= 1.7689922 \\ \text{diff.} &\quad 1817 \end{aligned}$$

$$\text{Diff. for } 60'' = 2892;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{1817}{2892} \times 60'' = 38'';$$

$$\therefore \frac{A-B}{2} = 30^\circ 26' 38'', \text{ and } \frac{A+B}{2} = 72^\circ 30';$$

$$\therefore A = 102^\circ 56' 38'', B = 42^\circ 3' 22''.$$

$$21. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11.29}{38.95} \cot 23^\circ 37' 30''.$$

$$\begin{aligned} \log \cot 23^\circ 37' &= .3592844 \\ \text{Subtract } \frac{30}{60} \times 3441 &\quad \quad 1721 \\ &\quad \quad \quad .3591123 \\ \log 11.29 &= 1.0526939 \\ &\quad 1.4118062 \\ \log 38.95 &= 1.5905075 \\ \log \tan \frac{B-C}{2} &= 1.8212987 \\ \log \tan 33^\circ 31' &= 1.8210574 \\ \text{diff.} &\quad 2413 \end{aligned}$$

$$\text{Diff. for } 60'' = 2743;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2413}{2743} \times 60'' = 53''.$$

$$\begin{aligned} \log \tan \frac{B-C}{2} &= 1.8212987 \\ \log \tan 33^\circ 31' &= 1.8210574 \\ \text{diff.} &\quad 2413 \end{aligned}$$

$$\therefore \frac{B-C}{2} = 33^\circ 31' 53'', \text{ and } \frac{B+C}{2} = 66^\circ 22' 30'';$$

$$\therefore B = 99^\circ 54' 23'', C = 32^\circ 50' 37''.$$

Again  $a = \frac{b \sin A}{\sin B} = \frac{25.12 \sin 47^\circ 15'}{\sin 99^\circ 54' 23''} = \frac{25.12 \sin 47^\circ 15'}{\cos 9^\circ 54' 23''}.$

$$\log \sin 47^\circ 15' = \bar{1}.8658868$$

$$\log 25.12 = 1.4000196$$

$$1.2659064$$

$$\bar{1}.9934760$$

$$\log a = \bar{1}.2724304$$

$$\log 18.725 = \bar{1}.2724218$$

$$86$$

$$4 \quad 93$$

$$\log \cos 9^\circ 54' = \bar{1}.9934844$$

$$\text{Subtract } \frac{23}{60} \times 220 = 84$$

$$\bar{1}.9934760$$

$$\therefore a = 18.7254.$$

$$22. \quad \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{1361.12 \times 1024.48}{1837.2 \times 2173.84}}.$$

$$\log 1361.1 = 3.1338900$$

$$2 \quad 64$$

$$\log 1024.4 = 3.0104696$$

$$8 \quad 339$$

$$6.1443999$$

$$6.6013840$$

$$2) \bar{1}.5430159$$

$$\log \sin \frac{B}{2} = \bar{1}.7715079$$

$$\log \sin 36^\circ 13' = \bar{1}.7714702$$

$$\text{diff.} \quad 377$$

$$\log 1837.2 = 3.2641564$$

$$\log 2173.8 = 3.3372196$$

$$4 \quad 80$$

$$6.6013840$$

$$\text{Diff. for } 60'' = 1724;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{377}{1724} \times 60'' = 13''.$$

$$\therefore \frac{B}{2} = 36^\circ 13' 13'', \text{ and } B = 72^\circ 26' 26''.$$

$$23. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{6.4405 \times 14.9114}{52.1248 \times 30.7728}}.$$

$$\log 6.4405 = .8089196$$

$$\log 14.911 = 1.1735068$$

$$4 \quad 116$$

$$1.9824380$$

$$3.2052113$$

$$2) \bar{2}.7772267$$

$$\log \tan \frac{A}{2} = \bar{1}.3886134$$

$$\log \tan 13^\circ 44' = \bar{1}.3880837$$

$$5297$$

$$\log 52.124 = 1.7170377$$

$$8 \quad 66$$

$$\log 30.772 = 1.4881557$$

$$8 \quad 113$$

$$3.2052113$$

$$\text{Diff. for } 60'' = 5475;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{5297}{5475} \times 60'' = 58''.$$

$$\therefore \frac{A}{2} = 13^\circ 44' 58'', \text{ and } A = 27^\circ 29' 56''.$$

Again  $\tan \frac{B}{2} = \sqrt{\frac{14.9114 \times 30.7728}{52.1248 \times 6.4405}}$

$$\begin{array}{rcl} \log 14.9114 & = & 1.1735184 \\ \log 30.7728 & = & 1.4881670 \\ & & 2.6616854 \\ & & 2.5259639 \\ 2) & \underline{.1357215} & \end{array}$$

$$\begin{array}{rcl} \log 52.1248 & = & 1.7170443 \\ \log 6.4405 & = & .8089196 \\ & & 2.5259639 \end{array}$$

$$\begin{array}{rcl} \log \tan \frac{B}{2} & = & .0678608 \\ \log \tan 49^\circ 27' & = & .0677338 \\ \text{diff.} & & 1270 \end{array}$$

$$\begin{array}{l} \text{Diff. for } 60'' = 2558; \\ \therefore \text{prop}^l. \text{ increase} = \frac{1270}{2558} \times 60'' = 30''. \end{array}$$

$$\therefore \frac{B}{2} = 49^\circ 27' 30'', \text{ and } B = 98^\circ 55'.$$

$$\therefore C = 53^\circ 35' 4''.$$

24.  $\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{202.949}{1497.597} \cot 51^\circ 36' 27''.$

$$\begin{array}{rcl} \log \cot 51^\circ 36' & = & \bar{1}.8990487 \\ \text{Subtract } \frac{27}{60} \times 2595 & = & 1168 \\ & & \bar{1}.8989319 \\ \log 202.94 & = & 2.3073677 \\ 9 & & 193 \\ & & 2.2063189 \\ & & 3.1753949 \\ \log \tan \frac{C-B}{2} & = & \bar{1}.0309240 \\ \log \tan 6^\circ 7' & = & \bar{1}.0300464 \\ \text{diff.} & & 8776 \end{array}$$

$$\begin{array}{rcl} \log 1497.5 & = & 3.1753668 \\ 9 & & 261 \\ 7 & & 203 \\ & & 3.1753949 \end{array}$$

$$\text{Diff. for } 60'' = 11909; \therefore \text{prop}^l. \text{ increase} = \frac{8776}{11909} \times 60'' = 44'';$$

$$\therefore \frac{C-B}{2} = 6^\circ 7' 44'', \text{ and } \frac{C+B}{2} = 38^\circ 23' 33'';$$

$$\therefore C = 44^\circ 31' 17'', B = 32^\circ 15' 49''.$$

To find  $a$ , we have  $a = \frac{b \sin A}{\sin B} = \frac{647.324 \sin 103^\circ 12' 54''}{\sin 32^\circ 15' 49''}$ .

Now  $\log \sin 103^\circ 12' 54'' = \log \sin 76^\circ 47' 6''$ .

$\begin{array}{r} \log \sin 76^\circ 47' = \bar{1}.9883415 \\ \frac{6}{60} \times 297 = \frac{30}{\bar{1}.9883445} \\ \log 647.32 = 2.8111190 \\ \quad 4 \quad \quad \quad 27 \\ \quad \quad \quad \hline \quad \quad \quad 2.7994662 \\ \quad \quad \quad \bar{1}.7273911 \\ \log a = 3.0720751 \\ \log 1180.5 = 3.0720660 \\ \quad \quad \quad \hline \quad \quad \quad 91 \\ \quad \quad \quad 2 \quad \quad \quad 73 \\ \quad \quad \quad \hline \quad \quad \quad 180 \\ \quad \quad \quad 5 \quad \quad \quad 185 \\ \quad \quad \quad \hline \end{array}$	$\begin{array}{r} \log \sin 32^\circ 15' = \bar{1}.7272276 \\ \frac{49}{60} \times 2002 = \frac{1635}{\bar{1}.7273911} \end{array}$
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$\therefore a = 1180.525$ .

25.

$a = \frac{b \sin A}{\sin B} = \frac{23.2783 \sin 37^\circ 57'}{\sin 43^\circ 13'}$ .

$\log 23.278$	$= 1.3669457$
3	56
$\log \sin 37^\circ 57'$	$= \bar{1}.7888565$
	$\bar{1}.1558078$
$\log \sin 43^\circ 13'$	$= \bar{1}.8355378$
$\log a$	$= 1.3202700$
$\log 20.905$	$= 1.3202502$
	$\hline 198$
9	$\hline 187$

$\therefore a = 20.9059$ .

Again 
$$c = \frac{b \sin C}{\sin B} = \frac{23.2783 \sin 81^\circ 10'}{\sin 43^\circ 13'}.$$

$$\begin{aligned} \log c &= 1.3669503 + \bar{1}.9948181 - \bar{1}.8355378 \\ &= 1.5262316 \\ \log 33.591 &= \underline{1.5262229} \\ &\quad 87 \\ &\quad 7 \quad \underline{90} \\ \therefore c &= 33.5917. \end{aligned}$$

26. 
$$b = \frac{c \sin B}{\sin C} = \frac{2484.3 \sin 72^\circ 43' 25''}{\sin 47^\circ 12' 17''}.$$

$$\begin{aligned} \log \sin 72^\circ 43' &= \bar{1}.9799339 \\ \frac{25}{60} \times 393 &= 164 \\ \log 2484.3 &= 3.3952040 \\ &\quad 3.3751543 \\ &\quad \bar{1}.8655693 \\ \log b &= 3.5095850 \\ \log 3232.8 &= \underline{3.5095788} \\ &\quad 62 \\ &\quad 4 \quad \underline{54} \\ &\quad 6 \quad \underline{8} \end{aligned}$$

$$\begin{aligned} \log \sin 47^\circ 12' &= \bar{1}.8655362 \\ \frac{17}{60} \times 1169 &= 331 \\ &\quad \bar{1}.8655693 \end{aligned}$$

$$\therefore b = 3232.846.$$

Again 
$$a = \frac{c \sin A}{\sin C} = \frac{2484.3 \sin 60^\circ 4' 18''}{\sin 47^\circ 12' 17''}.$$

$$\begin{aligned} \log \sin 60^\circ 4' &= \bar{1}.9378220 \\ \frac{18}{60} \times 727 &= 218 \\ \log 2484.3 &= 3.3952040 \\ &\quad 3.330478 \\ \log \sin 47^\circ 12' 17'' &= \bar{1}.8655693 \\ &\quad 3.4674785 \\ \log 2934.1 &= \underline{3.4674749} \\ &\quad 36 \\ &\quad 2 \quad \underline{30} \\ &\quad 4 \quad \underline{6} \end{aligned}$$

$$\therefore a = 2934.124.$$

$$27. \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{4367}{4667} \cot 15^\circ 45'.$$

$$\log \tan \frac{C-B}{2} = 3.6401832 + .5497060 - 3.6690378$$

$$= .5208514$$

$$\log \tan 73^\circ 13' = .5205681$$

$$\text{diff.} \quad \quad \quad 2833$$

$$\text{Diff. for } 60'' = 4568;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2833}{4568} \times 60'' = 37''.$$

$$\therefore \frac{C-B}{2} = 73^\circ 13' 37'', \text{ and } \frac{C+B}{2} = 74^\circ 15';$$

$$\therefore C = 147^\circ 28' 37'', B = 1^\circ 1' 23''.$$

Again

$$a = \frac{c \sin A}{\sin C} = \frac{4517 \sin 31^\circ 30'}{\sin 32^\circ 31' 23''}.$$

$$\log 4517 = 3.6548501$$

$$\log \sin 31^\circ 30' = 1.7180851$$

$$3.3729352$$

$$\log \sin 32^\circ 31' 23'' = 1.7304907$$

$$\log a = 3.6424445$$

$$\log 4389.8 = 3.6424447$$

$$\log \sin 32^\circ 31' = 1.7304148$$

$$\frac{23}{60} \times 1981 = 759$$

$$\log \sin 32^\circ 31' 23'' = 1.7304907$$

$$\therefore a = 4389.8 \text{ nearly.}$$

$$28. \sin A = \frac{a \sin C}{c} = \frac{324.68 \sin 35^\circ 17' 12''}{421.73}$$

$$\log \sin 35^\circ 17' = 1.7616424$$

$$\frac{12}{60} \times 1784 = 357$$

$$\log 324.68 = 2.5114555$$

$$2.2731336$$

$$\log 421.73 = 2.6250345$$

$$\log \sin A = 1.6480991$$

$$\log \sin 26^\circ 24' = 1.6480038$$

$$\text{diff.} \quad \quad \quad 953$$

$$\text{Diff. for } 60'' = 2544;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{953}{2544} \times 60'' = 23'';$$

$$\therefore A = 26^\circ 24' 23'', \text{ and } \therefore B = 118^\circ 18' 25''.$$

Again  $b = \frac{c \sin B}{\sin C} = \frac{421.73 \sin 61^\circ 41' 35''}{\sin 35^\circ 17' 12''}.$

$$\log \sin 61^\circ 41' = 1.9446501$$

$$\frac{35}{60} \times 680 = 397$$

$$\log 421.73 = 2.6250345$$

$$2.5697243$$

$$\log \sin 35^\circ 17' 12'' = 1.7616781$$

$$2.8080462$$

$$\log 642.75 = 2.8080421$$

$$41$$

$$6$$

$$41$$

$$\therefore b = 642.756.$$

29.  $\sin C = \frac{c \sin A}{a} = \frac{435.6 \sin 36^\circ 18' 27''}{321.7}.$

$$\log \sin 36^\circ 18' = 1.7723314$$

$$\frac{27}{60} \times 1719 = 774$$

$$\log 435.6 = 2.6390879$$

$$2.4114967$$

$$\log 321.7 = 2.5074511$$

$$\log \sin C = 1.9040456$$

$$\log \sin 53^\circ 17' = 1.9039587$$

$$\text{diff.} \quad 869$$

$$\text{Diff. for } 60'' = 943;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{869}{943} \times 60'' = 55'';$$

$$\therefore C = 53^\circ 17' 55'', \text{ or } 126^\circ 42' 5'', \text{ both values being admissible since } a < c.$$

30.  $\sin C = \frac{c \sin B}{b} = \frac{1665}{1325} \sin 52^\circ 19'.$

$$\log \sin C = 3.2214142 + 1.8983968 - 3.1222159$$

$$= 1.9975951$$

$$\log \sin 83^\circ 58' = 1.9975877$$

$$\text{diff.} \quad 74$$

$$\text{Diff. for } 60'' = 134;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{74}{134} \times 60'' = 33''.$$

$$\therefore C = 83^\circ 58' 33'', \text{ or } 96^\circ 1' 27'', \text{ both values being admissible since } b < c.$$

Now, with diagram of page 132, we have  $\angle BAC_2 = 31^\circ 39' 33''$ .

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{1325 \sin 31^\circ 39' 33''}{\sin 52^\circ 19'}$$

$$\log \sin 31^\circ 39' = \bar{1}.7199350$$

$$\frac{33}{60} \times 2049 = 1127$$

$$\log 1325 = 3.1222159$$

$$\hline 2.8422636$$

$$\log \sin 52^\circ 19' = \bar{1}.8983968$$

$$\log a = 2.9438668$$

$$\log 878.75 = 2.9438653$$

$$\begin{array}{r} 15 \\ 3 \quad 15 \\ \hline \end{array}$$

$$\therefore a = 878.753.$$

$$31. \text{ Here } A = 64^\circ 26' 15'', \quad b = \frac{a \sin B}{\sin A} = \frac{3795 \sin 73^\circ 15' 15''}{\sin 64^\circ 26' 15''}.$$

$$\log \sin 73^\circ 15' = \bar{1}.9811711$$

$$\frac{15}{60} \times 380 = 95$$

$$\log 3795 = 3.5792118$$

$$\hline 3.5603924$$

$$\log \sin 64^\circ 26' 15'' = \bar{1}.9552620$$

$$\log 6 = 3.6051304$$

$$\log 4028.3 = 3.6051218$$

$$\begin{array}{r} 86 \\ 8 \quad 86 \\ \hline \end{array}$$

$$\log \sin 64^\circ 26' = \bar{1}.9552469$$

$$\frac{15}{60} \times 604 = 151$$

$$\hline \bar{1}.9552620$$

$$\therefore b = 4028.38.$$

Again

$$c = \frac{a \sin C}{\sin A} = \frac{3795 \sin 42^\circ 18' 30''}{\sin 64^\circ 26' 15''}.$$

$$\log 3795 = 3.5792118$$

$$\log \sin 42^\circ 18' = \bar{1}.8280231$$

$$\frac{30}{60} \times 1388 = 694$$

$$\hline 3.4073043$$

$$\log \sin A = \bar{1}.9552620$$

$$\log c = 3.4520423$$

$$\log 2831.6 = 3.4520319$$

$$\begin{array}{r} 104 \\ 7 \quad 107 \\ \hline \end{array}$$

$$\therefore c = 2831.67.$$

$$32. \sin B = \frac{b}{c} \sin C = \frac{17}{12} \sin 43^\circ 12' 12''.$$

$$\log 17 = 1.2304489$$

$$\log \sin 43^\circ 12' = 1.8354033$$

$$\frac{12}{60} \times 1345 = 269$$

$$\underline{1.0658791}$$

$$\log 12 = 1.0791812$$

$$\log \sin B = 1.9866979$$

$$\log \sin 75^\circ 53' = 1.9866827$$

$$\text{diff.} \quad \underline{152}$$

$$\text{Diff. for } 60'' = 317;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{152}{317} \times 60'' = 29''.$$

$$\therefore B = 75^\circ 53' 29'', \text{ or } 104^\circ 6' 31'', \text{ both values being admissible since } c < b.$$

$$\therefore A = 60^\circ 54' 19'', \text{ or } 32^\circ 41' 17''.$$

$$33. \text{ Let } b = 2.7402, c = .7401, A = 59^\circ 27' 5''.$$

$$\tan \frac{B-C}{2} = \frac{2.0001}{3.4803} \cot 29^\circ 43' 32.5''.$$

$$\log \cot 29^\circ 43' = .2435347$$

$$\frac{32.5}{60} \times 2934 = 1589$$

$$\log 2.0001 = .3010517$$

$$\underline{.5444275}$$

$$\log 3.4803 = .5416167$$

$$\log \tan \frac{B-C}{2} = .0028108$$

$$\log \tan 45^\circ 11' = .0027793$$

$$\text{diff.} \quad \underline{315}$$

$$\text{Diff. for } 60'' = 2527;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{315}{2527} \times 60'' = 7.5''.$$

$$\therefore \frac{B-C}{2} = 45^\circ 11' 7.5'', \text{ and } \frac{B+C}{2} = 60^\circ 16' 27.5'';$$

$$\therefore B = 105^\circ 27' 35'', C = 15^\circ 5' 20''.$$

Again

$$a = \frac{c \sin A}{\sin C} = \frac{.7401 \sin 59^\circ 27' 5''}{\sin 15^\circ 5' 20''}.$$

$$\log \sin 59^\circ 27' = 1.9350969$$

$$\frac{5}{60} \times 746 = 62$$

$$\log .7401 = 1.8692904$$

$$\underline{1.8043935}$$

$$\underline{1.4155029}$$

$$\log a = .3888906$$

$$\log 2.4484 = .3888824$$

$$\underline{82}$$

$$5 \quad \underline{89}$$

$$\log \sin 15^\circ 5' = 1.4153468$$

$$\frac{20}{60} \times 4684 = 1561$$

$$\underline{1.4155029}$$

$$\therefore a = 2.44845.$$

Let  $h$  = the altitude; then  $h = b \sin C$ ;

$$\begin{aligned}\therefore \log h &= \log 2.7402 + \log \sin 15^\circ 5' 20'' \\ &= .4379823 + \bar{1}.4155029 \\ &= \bar{1}.8532852 \\ \log .71332 &= \bar{1}.8532844 \\ &\quad \begin{array}{r} 8 \\ 1 \quad 6 \\ - \end{array} \\ \therefore \text{altitude} &= .713321.\end{aligned}$$

34. Let  $b = 105.25$ ,  $c = 76.75$ ,  $B - C = 17^\circ 48''$ ;

then  $\cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2} = \frac{182}{28.5} \tan 8^\circ 54'.$

$$\begin{aligned}\log \cot \frac{A}{2} &= \log 182 + \log \tan 8^\circ 54' - \log 28.5 \\ &= 2.2600714 + \bar{1}.1947802 - 1.4548449 \\ &= .0000067 = \log \cot 45^\circ \text{ nearly.} \\ \therefore A &= 90^\circ \text{ nearly.}\end{aligned}$$

35. (1)  $\sin C = \frac{c \sin A}{a} = \frac{36.5 \sin 43^\circ 15'}{20};$

$$\begin{aligned}\log \sin C &= 1.5622929 + \bar{1}.8358066 - 1.3010300 \\ &= .0970695,\end{aligned}$$

which is impossible, since  $\sin C$  must be  $< 1$ .

(2)  $\sin C = \frac{36.5 \sin 43^\circ 15'}{30};$

$$\begin{aligned}\log \sin 43^\circ 15' &= \bar{1}.8358066 \\ \log 36.5 &= 1.5622929 \\ &\quad \underline{1.3980995} \\ \log 30 &= 1.4771213 \\ \log \sin C &= \bar{1}.9209782\end{aligned}$$

Thus  $C$  is not a right angle, and since  $a < c$  the solution is ambiguous.

(3)  $\sin C = \frac{36.5 \sin 43^\circ 15'}{45}.$

$$\begin{aligned}\log \sin C &= 1.3980995 - 1.6532125 \\ &= \bar{1}.7448870\end{aligned}$$

$$\begin{aligned}\log \sin 33^\circ 45' &= \bar{1}.7447390 \\ \text{diff.} &\quad \underline{1480}\end{aligned}$$

Diff. for  $60'' = 1890$ ;

$$\therefore \text{prop}^l. \text{ increase} = \frac{148}{189} \times 60'' = 47''.$$

$$\therefore C = 33^\circ 45' 47''; \therefore B = 102^\circ 59' 13''.$$

$$\text{Now } b = \frac{a \sin B}{\sin A} = \frac{45 \cos 12^\circ 59' 13''}{\sin 43^\circ 15'};$$

$$\log \cos 12^\circ 59' = \bar{1}.9887531$$

$$\text{Subtract } \frac{13}{60} \times 292 = 63$$

$$\bar{1}.9887468$$

$$\log 45 = 1.6532125$$

$$\bar{1}.6419593$$

$$\log \sin 43^\circ 15' = \bar{1}.8358066$$

$$\bar{1}.8061527$$

$$\log 63.996 = 1.8061528$$

$$\text{Thus } b = 63.996.$$

36. For the first part of the Example, see Art. 197.

$$\tan \theta = \frac{2 \sqrt{17.32 \times 13.47}}{3.85} \sin 23^\circ 36' 30'';$$

$$\log \sin 23^\circ 36' = \bar{1}.6024388$$

$$\frac{30}{60} \times 2890 = 1445$$

$$\log 2 = .3010300$$

$$1.1839578$$

$$\bar{1}.0875711$$

$$\log 3.85 = .5854607$$

$$\log \tan \theta = .5021104$$

$$\log \tan 72^\circ 31' = .5017184$$

$$3920$$

$$\therefore \theta = 72^\circ 31' 53''.$$

$$\log 17.32 = 1.2385479$$

$$\log 13.47 = 1.1293676$$

$$2) 2.3679155$$

$$1.1839578$$

$$\text{Diff. for } 60'' = 4410;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{392}{441} \times 60'' = 53''.$$

$$\text{Again } c = (a - b) \sec \theta;$$

$$\log 3.85 = .5854607$$

$$\log \sec 72^\circ 31' = .5222591$$

$$\frac{53}{60} \times 4013 = 3545$$

$$\bar{1}.1080743$$

$$\log 12.825 = 1.1080574$$

$$169$$

$$5$$

$$170$$

$$\therefore c = 12.8255.$$

37. See Art. 195.

$$\tan \phi = \frac{44.1}{10.5} \tan 22^\circ 36';$$

$$\log 44.1 = 1.6444386$$

$$\log \tan 22^\circ 36' = 1.6193645$$

$$\hline 1.2638031$$

$$\log 10.5 = 1.0211893$$

$$\log \tan \phi = .2426138$$

$$\log \tan 60^\circ 13' = .2423617$$

$$\text{diff.} \quad \hline 2521$$

$$\text{Diff. for } 60'' = 2930;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2521}{2930} \times 60'' = 52''.$$

$$\therefore \phi = 60^\circ 13' 52''.$$

$$\text{Again } c = (a - b) \cos \frac{C}{2} \sec \phi;$$

$$\log 10.5 = 1.0211893$$

$$\log \cos 22^\circ 36' = 1.9653006$$

$$\log \sec 60^\circ 13' = .3038870$$

$$\frac{52}{60} \times 2208 = 1914$$

$$\log c = 1.2905683$$

$$\log 19.523 = 1.2905466$$

$$\hline 217$$

$$9 \quad \hline 200$$

$$\hline 170$$

$$8 \quad \hline 178$$

$$\therefore c = 19.52398.$$

## EXAMPLES. XVI. g. PAGE 183 D.

$$1. \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{16.3 \times 9}{11.7 \times 19}}.$$

$$\log 16.3 = 1.2122$$

$$\log 9 = .9542$$

$$\hline 2.1664$$

$$2.3469$$

$$2) \hline 1.8195$$

$$\log 11.7 = 1.0682$$

$$\log 19 = 1.2787$$

$$\hline 2.3469$$

$$\log \sin \frac{A}{2} = 1.9097; \text{ whence } \frac{A}{2} = 54^\circ 19'.$$

$$\therefore A = 108^\circ 38'.$$

$$2. \quad \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{112.5 \times 19.25}{68.75 \times 63}}.$$

$$\begin{array}{r} \log 112.5 = 2.0511 \\ \log 19.25 = 1.2844 \\ \hline 3.3355 \end{array}$$

$$\begin{array}{r} \log 68.75 = 1.8373 \\ \log 63 = 1.7993 \\ \hline 3.6366 \end{array}$$

$$\begin{array}{r} 3.3355 \\ 3.6366 \\ \hline 2) 1.6989 \end{array}$$

$$\log \cos \frac{B}{2} = 1.8495; \text{ whence } \frac{B}{2} = 45^\circ.$$

$$\therefore B = 90^\circ.$$

$$3. \quad \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{5.72 \times 3.31}{11.24 \times 13.65}}.$$

$$\begin{array}{r} \log 5.72 = .7572 \\ \log 3.31 = .5198 \\ \hline 1.2772 \end{array}$$

$$\begin{array}{r} \log 11.24 = 1.0508 \\ \log 13.65 = 1.1351 \\ \hline 2.1859 \end{array}$$

$$\begin{array}{r} 1.2772 \\ 2.1859 \\ \hline 2) 1.0913 \end{array}$$

$$\log \sin \frac{C}{2} = 1.5457; \text{ whence } \frac{C}{2} = 20^\circ 31'.$$

$$\therefore C = 41^\circ 8'.$$

$$4. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{1 \times 14}{23 \times 8}}.$$

$$\begin{array}{r} \log 14 = 1.1461 \\ 2.2648 \end{array}$$

$$\begin{array}{r} \log 23 = 1.3617 \\ \log 8 = .9031 \\ \hline 2.2648 \end{array}$$

$$2) 2.8813$$

$$\log \tan \frac{A}{2} = 1.4407; \text{ whence } \frac{A}{2} = 15^\circ 25'.$$

$$\therefore A = 30^\circ 50'.$$

$$\text{Again, } \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{8 \times 14}{23 \times 1}}.$$

$$\begin{array}{r} \log 8 = .9031 \\ \log 14 = 1.1461 \\ \hline 2.0492 \end{array}$$

$$\log 23 = 1.3617$$

$$\begin{array}{r} 2.0492 \\ 1.3617 \\ \hline 2) 0.6875 \end{array}$$

$$\log \tan \frac{B}{2} = .3438$$

$$\begin{array}{r} \log \tan 65^\circ 36' = .3433 \\ \text{diff.} \quad 5 \end{array}$$

Since 5 is the mean of the differences 3 and 7, the corresponding increase in the angle is 1'.5.

$$\text{Hence } \frac{B}{2} = 65^\circ 37'.5, \text{ and } B = 131^\circ 15'.$$

$$\therefore C = 17^\circ 55'.$$

$$5. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{24.22 \times 7.32}{70.72 \times 39.18}}$$

$\log 24.22 = 1.3842$ $\log 7.32 = .8645$ $\hline 2.2487$ $3.4426$ $2) \overline{2.8061}$	$\log 70.72 = 1.8495$ $\log 39.18 = 1.5931$ $\hline 3.4426$
---	---

$$\tan \frac{A}{2} = 1.4031; \text{ whence } \frac{A}{2} = 14^\circ 12'. \quad \therefore A = 28^\circ 24'.$$

Again,  $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{39.18 \times 7.32}{70.72 \times 24.22}}$

$\log 39.18 = 1.5931$ $\log 7.32 = .8645$ $\hline 2.4576$ $3.2337$ $2) \overline{1.2239}$	$\log 70.72 = 1.8495$ $\log 24.22 = 1.3842$ $\hline 3.2337$
---	---

$$\log \tan \frac{B}{2} = 1.6120; \text{ whence } \frac{B}{2} = 22^\circ 15'.$$

$$\therefore B = 44^\circ 30', \text{ and } C = 107^\circ 6'.$$

$$6. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{11.02 \times 22.38}{82.62 \times 49.22}}$$

$\log 11.02 = 1.0422$ $\log 22.38 = 1.3498$ $\hline 2.3920$ $3.6093$ $2) \overline{2.7827}$	$\log 82.62 = 1.9171$ $\log 49.22 = 1.6922$ $\hline 3.6093$
---	---

$$\tan \frac{A}{2} = 1.3914; \text{ whence } \frac{A}{2} = 13^\circ 50'. \quad \therefore A = 27^\circ 40'.$$

Again,  $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{49.22 \times 22.38}{82.62 \times 11.02}}$

$\log 49.22 = 1.6922$ $\log 22.38 = 1.3498$ $\hline 3.0420$ $2.9593$ $2) \overline{0.0827}$	$\log 82.62 = 1.9171$ $\log 11.02 = 1.0422$ $\hline 2.9593$
---	---

$$\tan \frac{B}{2} = .0414; \text{ whence } \frac{B}{2} = 47^\circ 43'.5, \text{ as in Ex. 4 above.}$$

$$\therefore B = 95^\circ 27'; \text{ and } C = 56^\circ 53'.$$

$$7. \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{15.98 \times 1.23}{6.84 \times 10.37}}$$

$\log 15.98 = 1.2036$ $\log 1.23 = .0899$ $\hline 1.2935$ $1.8508$ $2) \overline{1.4427}$	$\log 6.84 = .8351$ $\log 10.37 = 1.0157$ $\hline 1.8508$
---	---

$\log \cos \frac{A}{2} = \bar{1}.7214$ ; whence  $\frac{A}{2} = 58^\circ 14'$ .  $\therefore A = 116^\circ 28'$ .

$$8. \quad \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{10.85 \times 2.85}{21.5 \times 29.5}}$$

$\log 10.85 = 1.0355$ $\log 2.85 = .4548$ $\hline 1.4903$ $2.8022$ $2) \overline{2.6881}$	$\log 21.5 = 1.3324$ $\log 29.5 = 1.4698$ $\hline 2.8022$
---	---

$\log \sin \frac{B}{2} = \bar{1}.3441$ ; whence  $\frac{B}{2} = 12^\circ 45'.5$ , as in Ex. 4 above.  
 $\therefore B = 25^\circ 31'$ .

$$9. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{3 \times 2}{10 \times 5}} = \sqrt{.12}$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{5 \times 2}{10 \times 3}} = \sqrt{\frac{1}{3}}$$

$\therefore \log \tan \frac{A}{2} = \frac{1}{2} (\bar{1}.0792) = \bar{1}.5396$ ;  
 whence  $\frac{A}{2} = 19^\circ 6'$ .

$\therefore \log \tan \frac{B}{2} = \frac{1}{2} (-\log 3) = \frac{1}{2} (\bar{1}.5229) = \bar{1}.7614$ ;  
 whence  $\frac{B}{2} = 30^\circ$ .

Thus  $A = 38^\circ 12'$ ,  $B = 60^\circ$ ,  $C = 81^\circ 48'$ .

$$10. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{7 \times 6}{21 \times 8}}$$

$\log 7 = .8451$ $\log 6 = .7782$ $\hline 1.6233$ $2.2253$ $2) \overline{1.3980}$	$\log 21 = 1.3222$ $\log 8 = .9031$ $\hline 2.2253$
---	---

$\tan \frac{A}{2} = \bar{1}.6990$ ; whence  $\frac{A}{2} = 26^\circ 34'$ .  $\therefore A = 53^\circ 8'$ .

$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{8 \times 6}{21 \times 7}}$$

$$\log 8 = .9031$$

$$\log 6 = .7782$$

$$\underline{1.6813}$$

$$\underline{2.1673}$$

$$2) \underline{1.5140}$$

$$\log 21 = 1.3222$$

$$\log 7 = .8451$$

$$\underline{2.1673}$$

$$\log \tan \frac{B}{2} = \bar{1}.7570; \text{ whence } \frac{B}{2} = 29^\circ 45'.$$

$$\therefore B = 59^\circ 30', \text{ and } C = 67^\circ 22'.$$

$$11. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{17.8}{47.8} \cot 53^\circ 43'.$$

$$\log 17.8 = 1.2504$$

$$\log \cot 53^\circ 43' = \bar{1}.8658$$

$$\underline{1.1162}$$

$$\log 47.8 = 1.6794$$

$$\log \tan \frac{B-C}{2} = \bar{1}.4368;$$

$$\frac{B+C}{2} = 36^\circ 17',$$

$$\text{whence } \frac{B-C}{2} = 15^\circ 18'.$$

$$\therefore B = 51^\circ 35', \text{ and } C = 20^\circ 59'.$$

$$12. \quad \tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{38.7}{232.1} \cot 61^\circ 51'.$$

$$\log 38.7 = 1.5877$$

$$\log \cot 61^\circ 51' = \bar{1}.7224$$

$$\underline{1.3101}$$

$$\log 232.1 = 2.3657$$

$$\log \tan \frac{B-A}{2} = \bar{2}.9504;$$

$$\frac{B+A}{2} = 28^\circ 9',$$

$$\text{whence } \frac{B-A}{2} = 5^\circ 6'.$$

$$\therefore B = 33^\circ 15', \text{ and } A = 23^\circ 3'.$$

$$13. \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{.564}{.588} \cot 30^\circ 15'.$$

$$\log 564 = 2.7513$$

$$\log \cot 30^\circ 15' = .2342$$

$$\underline{2.9855}$$

$$\log 588 = 2.7694$$

$$\log \tan \frac{C-A}{2} = .2161;$$

$$\frac{C+A}{2} = 59^\circ 45',$$

$$\text{whence } \frac{C-A}{2} = 58^\circ 42'.$$

$$\therefore C = 118^\circ 27', \text{ and } A = 1^\circ 3'.$$

14. Here  $a = 27.3$ ,  $b = 16.8$ ,  $C = 45^\circ 7'$ . Required  $A$ ,  $B$ , and  $c$ .

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{10.5}{44.1} \cot 22^\circ 33'.5.$$

$$\begin{aligned} \log 10.5 &= 1.0212 \\ \log \cot 22^\circ 33'.5 &= .3815 \\ \hline &1.4027 \end{aligned}$$

$$\log 44.1 = 1.6444$$

$$\log \tan \frac{A-B}{2} = 1.7583;$$

$$\frac{A+B}{2} = 67^\circ 26'.5,$$

$$\text{whence } \frac{A-B}{2} = 29^\circ 49'.$$

$$\therefore A = 97^\circ 15'.5, \text{ and } B = 37^\circ 37'.5.$$

Again,

$$c = \frac{b \sin C}{\sin B} = \frac{16.8 \sin 45^\circ 7'}{\sin 37^\circ 37'.5}.$$

$$\begin{aligned} \log 16.8 &= 1.2253 \\ \log \sin 45^\circ 7' &= 1.8503 \\ \hline &1.0756 \end{aligned}$$

$$\log \sin 37^\circ 37'.5 = 1.7857$$

$$\log c = 1.2899; \text{ whence } c = 19.49.$$

15.

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{57.8}{108.0} \cot 18^\circ 30'.$$

$$\begin{aligned} \log 57.8 &= 1.7619 \\ \log \cot 18^\circ 30' &= .4755 \\ \hline &2.2374 \end{aligned}$$

$$\log 108 = 2.0334$$

$$\log \tan \frac{B-C}{2} = .2040;$$

$$\frac{B+C}{2} = 71^\circ 30',$$

$$\text{whence } \frac{B-C}{2} = 57^\circ 59'.$$

$$\therefore B = 129^\circ 29', \text{ and } C = 13^\circ 31'.$$

Again,

$$a = \frac{c \sin A}{\sin C} = \frac{25.1 \sin 37^\circ}{\sin 13^\circ 31'}.$$

$$\begin{aligned} \log 25.1 &= 1.3997 \\ \log \sin 37^\circ &= 1.7795 \\ \hline &1.1792 \end{aligned}$$

$$\log \sin 13^\circ 31' = 1.3687$$

$$\log a = 1.8105; \text{ whence } a = 64.65.$$

16.

$$\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{6.48}{60.48} \cot 30^\circ.$$

$$\begin{aligned} \log 6.48 &= .8116 \\ \log \cot 30^\circ &= .2386 \\ \hline &1.0502 \end{aligned}$$

$$\log 60.48 = 1.7816$$

$$\log \tan \frac{C-B}{2} = 1.2686;$$

$$\frac{C+B}{2} = 60^\circ,$$

$$\text{whence } \frac{C-B}{2} = 10^\circ 31'.$$

$$\therefore C = 70^\circ 31', \text{ and } B = 49^\circ 29'.$$

$$17. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{8}{22} \cot 41^\circ 7' = \frac{4}{11} \cot 41^\circ 7'.$$

$$\begin{array}{r} \log 4 = .6021 \\ \log \cot 41^\circ 7' = .0590 \\ \hline .6611 \end{array}$$

$$\log 11 = 1.0414$$

$$\log \tan \frac{A-B}{2} = \bar{1}.6197;$$

$$\frac{A+B}{2} = 48^\circ 53',$$

$$\text{whence } \frac{A-B}{2} = 22^\circ 37'.$$

$$\therefore A = 71^\circ 30', \text{ and } B = 26^\circ 16'.$$

$$18. \quad \text{Let } a = 2b, \text{ and } C = 52^\circ 47', \text{ then}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{3} \cot 26^\circ 23' \cdot 5$$

$$= \frac{1}{3} \times 2.0152 = .6717 = \tan 33^\circ 53'.$$

$$\therefore \frac{A+B}{2} = 63^\circ 36' \cdot 5, \text{ and } \frac{A-B}{2} = 33^\circ 53'.$$

$$\therefore A = 97^\circ 29' \cdot 5, \text{ and } B = 29^\circ 43' \cdot 5.$$

$$19. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{44}{218} \cot 9^\circ 8'.$$

$$\begin{array}{r} \log 44 = 1.6435 \\ \log \cot 9^\circ 8' = .7938 \\ \hline 2.4373 \end{array}$$

$$\log 218 = 2.3385$$

$$\log \tan \frac{B-C}{2} = .0988;$$

$$\frac{B+C}{2} = 80^\circ 52',$$

$$\text{whence } \frac{B-C}{2} = 51^\circ 28'.$$

$$\therefore B = 132^\circ 20', \text{ and } C = 29^\circ 24'.$$

$$20. \quad \tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{6.43}{41.63} \cot 60^\circ 49'.$$

$$\begin{array}{r} \log 6.43 = .8082 \\ \log \cot 60^\circ 49' = \bar{1}.7470 \\ \hline .5552 \end{array}$$

$$\log 41.63 = 1.6194$$

$$\log \tan \frac{B-A}{2} = \bar{2}.9358;$$

$$\frac{B+A}{2} = 29^\circ 11',$$

$$\text{whence } \frac{B-A}{2} = 4^\circ 56'.$$

$$\therefore B = 34^\circ 7', \text{ and } A = 24^\circ 15'.$$

$$\text{Again, } c = \frac{a \sin C}{\sin A} = \frac{17.6 \sin 121^\circ 38'}{\sin 24^\circ 15'} = \frac{17.6 \sin 58^\circ 22'}{\sin 24^\circ 15'}.$$

$$\log 17.6 = 1.2455$$

$$\log \sin 58^\circ 22' = \bar{1}.9301$$

$$\hline 1.1755$$

$$\log \sin 24^\circ 15' = \bar{1}.6135$$

$$\log c = 1.5620; \text{ whence } c = 36.48.$$

$$21. \quad a = \frac{b \sin A}{\sin B} = \frac{100 \sin 40^\circ}{\sin 70^\circ}.$$

$$\log 100 = 2$$

$$\log \sin 40^\circ = \bar{1}.8081$$

$$\log \sin 70^\circ = \bar{1}.9730$$

$$\log a = 1.8351; \text{ whence } a = 68.41.$$

$$22. \quad b = \frac{a \sin B}{\sin A} = \frac{85.2 \sin 42^\circ}{\sin 31^\circ}.$$

$$\log 85.2 = 1.9304$$

$$\log \sin 42^\circ = \bar{1}.8255$$

$$\log \sin 31^\circ = \bar{1}.7118$$

$$\log b = 2.0441; \text{ whence } b = 110.7.$$

$$23. \quad a = \frac{c \sin A}{\sin C} = \frac{5.23 \sin 49^\circ 11'}{\sin 109^\circ 34'} = \frac{5.23 \sin 49^\circ 11'}{\sin 70^\circ 26'}.$$

$$\log 5.23 = .7185$$

$$\log \sin 49^\circ 11' = \bar{1}.8789$$

$$\log \sin 70^\circ 26' = \bar{1}.9742$$

$$\log a = .6232; \text{ whence } a = 4.200.$$

$$24. \quad c = \frac{b \sin C}{\sin B} = \frac{873 \sin 71^\circ 35'}{\sin 42^\circ 58'}.$$

$$\log 873 = 2.9410$$

$$\log \sin 71^\circ 35' = \bar{1}.9772$$

$$\log \sin 42^\circ 58' = \bar{1}.8336$$

$$\log c = 3.0846; \text{ whence } c = 1215.$$

$$25. \quad a = \frac{c \sin A}{\sin C} = \frac{60 \sin 60^\circ}{\sin 40^\circ 40'}.$$

$$\log 60 = 1.7782$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log \sin 40^\circ 40' = \bar{1}.8140$$

$$\log a = 1.9017; \text{ whence } a = 79.75.$$

$$26. \quad a = \frac{c \sin A}{\sin C} = \frac{3.57 \sin 51^\circ 51'}{\sin 40^\circ 26'}.$$

$$\log 3.57 = .5527$$

$$\log \sin 51^\circ 51' = \bar{1}.8956$$

$$\underline{.4483}$$

$$\log \sin 40^\circ 26' = \bar{1}.8120$$

$$\log a = .6363$$

$$\therefore a = 4.328.$$

$$b = \frac{c \sin B}{\sin C} = \frac{3.57 \sin 87^\circ 43'}{\sin 40^\circ 26'}.$$

$$\log 3.57 = .5527$$

$$\log \sin 87^\circ 43' = \bar{1}.9996$$

$$\underline{.5523}$$

$$\log \sin 40^\circ 26' = \bar{1}.8120$$

$$\log b = .7403$$

$$\therefore b = 5.499.$$

27.

$$b = \frac{a \sin B}{\sin A} = \frac{125.7 \sin 65^\circ 47'}{\sin 61^\circ 34'}.$$

$$\log 125.7 = 2.0993$$

$$\log \sin 65^\circ 47' = \bar{1}.9600$$

$$\underline{2.0593}$$

$$\log \sin 61^\circ 34' = \bar{1}.9442$$

$$\log b = 2.1151; \text{ whence } b = 130.3.$$

$$28. \quad a = \frac{c \sin A}{\sin C} \\ = \frac{92.93 \sin 72^\circ 19'}{\sin 24^\circ 24'}.$$

$$\log 92.93 = 1.9681$$

$$\log \sin 72^\circ 19' = \bar{1}.9789$$

$$\underline{1.9470}$$

$$\log \sin 24^\circ 24' = \bar{1}.6161$$

$$\log a = 2.3309$$

$$\therefore a = 214.2.$$

$$b = \frac{c \sin B}{\sin C} \\ = \frac{92.93 \sin 83^\circ 17'}{\sin 24^\circ 24'}.$$

$$\log 92.93 = 1.9681$$

$$\log \sin 83^\circ 17' = \bar{1}.9970$$

$$\underline{1.9651}$$

$$\log \sin 24^\circ 24' = \bar{1}.6161$$

$$\log b = 2.3490$$

$$\therefore b = 223.4.$$

$$29. \quad b = \frac{a \sin B}{\sin A} \\ = \frac{4.375 \sin 49^\circ 30'}{\sin 60^\circ}.$$

$$\log 4.375 = .6410$$

$$\log \sin 49^\circ 30' = \bar{1}.8810$$

$$\underline{.5220}$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log b = .5845$$

$$\therefore b = 3.841.$$

$$c = \frac{a \sin C}{\sin A} \\ = \frac{4.375 \sin 70^\circ 30'}{\sin 60^\circ}.$$

$$\log 4.375 = .6410$$

$$\log \sin 70^\circ 30' = \bar{1}.9743$$

$$\underline{.6153}$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log c = .6778$$

$$\therefore c = 4.762.$$

$$30. \quad \frac{A}{1} = \frac{B}{4} = \frac{C}{7} = \frac{A+B+C}{12} = \frac{180^\circ}{12} = 15^\circ.$$

$$\therefore A = 15^\circ, B = 60^\circ, C = 105^\circ.$$

$$a = \frac{b \sin A}{\sin B} = \frac{89.36 \sin 15^\circ}{\sin 60^\circ}.$$

$$\log 89.36 = 1.9512$$

$$\log \sin 15^\circ = \bar{1}.4130$$

$$\hline 1.3642$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log a = \bar{1}.4267$$

$$\therefore a = 26.71.$$

$$c = \frac{b \sin C}{\sin B} = \frac{89.36 \sin 75^\circ}{\sin 60^\circ}.$$

$$\log 89.36 = 1.9512$$

$$\log \sin 75^\circ = \bar{1}.9849$$

$$\hline 1.9361$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log c = \bar{1}.9986$$

$$\therefore c = 99.68.$$

$$31. \quad \sin B = \frac{b \sin A}{a} = \frac{62 \sin 82^\circ 14'}{73}.$$

$$\log 62 = 1.7924$$

$$\log \sin 82^\circ 14' = \bar{1}.9960$$

$$\hline 1.7884$$

$$\log 73 = 1.8633$$

$$\log \sin B = \bar{1}.9251; \text{ whence } B = 57^\circ 18'.$$

$$32. \quad \sin C = \frac{c \sin B}{b} = \frac{63.45 \sin 27^\circ 15'}{41.62}.$$

$$\log 63.45 = 1.8024$$

$$\log \sin 27^\circ 15' = \bar{1}.6607$$

$$\hline 1.4631$$

$$\log 41.62 = 1.6193$$

$$\log \sin C = \bar{1}.8438;$$

whence  $C = 44^\circ 16'$ , or  $135^\circ 44'$ , both values being admissible since  $b < c$ .

$$33. \quad \sin A = \frac{a \sin B}{b} = \frac{17.28 \sin 55^\circ 13'}{23.97}.$$

$$\log 17.28 = 1.2375$$

$$\log \sin 55^\circ 13' = \bar{1}.9145$$

$$\hline 1.1520$$

$$\log 23.97 = 1.3797$$

$$\log \sin A = \bar{1}.7723$$

$$\therefore A = 36^\circ 18'$$

$$\text{and } C = 88^\circ 29'.$$

$$c = \frac{a \sin C}{\sin A} = \frac{17.28 \sin 88^\circ 29'}{\sin 36^\circ 18'}.$$

$$\log 17.28 = 1.2375$$

$$\log \sin 88^\circ 29' = \bar{1}.9999$$

$$\hline 1.2374$$

$$\log \sin 36^\circ 18' = \bar{1}.7723$$

$$\log c = 1.4651$$

$$\therefore c = 29.18.$$

$$34. \quad \sin B = \frac{b \sin A}{a} = \frac{141.3 \sin 40^\circ}{94.2}.$$

$\log \sin B = 2.1501 + \bar{1}.8081 - 1.9741 = \bar{1}.9841$ ; whence  $B = 74^\circ 36'$ , or  $105^\circ 24'$ , since  $a < b$ .  $\therefore C_1 = 65^\circ 24'$  and  $C_2 = 34^\circ 36'$ .

$c_1 = \frac{a \sin C_1}{\sin A} = \frac{94.2 \sin 65^\circ 24'}{\sin 40^\circ}.$ $\log 94.2 = 1.9741$ $\log \sin 65^\circ 24' = \bar{1}.9587$ $\hline 1.9328$ $\log \sin 40^\circ = \bar{1}.8081$ $\log c_1 = \bar{2}.1247$ $\therefore c_1 = 133.2.$	$c_2 = \frac{a \sin C_2}{\sin A} = \frac{94.2 \sin 34^\circ 36'}{\sin 40^\circ}.$ $\log 94.2 = 1.9741$ $\log \sin 34^\circ 36' = \bar{1}.7542$ $\hline 1.7283$ $\log \sin 40^\circ = \bar{1}.8081$ $\log c_2 = \bar{1}.9202$ $\therefore c_2 = 83.22.$
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$$35. \quad \sin B = \frac{b \sin A}{a} = \frac{137 \sin 20^\circ 41'}{115}.$$

$\therefore \log \sin B = 2.1367 + \bar{1}.5481 - 2.0607 = \bar{1}.6241$ ; whence  $B = 24^\circ 53'$ , or  $155^\circ 7'$ , since  $a < b$ .  $\therefore C_1 = 134^\circ 26'$  and  $C_2 = 4^\circ 12'$ .

$c_1 = \frac{a \sin C_1}{\sin A} = \frac{115 \sin 45^\circ 34'}{\sin 20^\circ 41'}.$ $\log 115 = 2.0607$ $\log \sin 45^\circ 34' = \bar{1}.8538$ $\hline 1.9145$ $\log \sin 20^\circ 41' = \bar{1}.5481$ $\log c_1 = \bar{2}.3664$ $\therefore c_1 = 232.5.$	$c_2 = \frac{a \sin C_2}{\sin A} = \frac{115 \sin 4^\circ 12'}{\sin 20^\circ 41'}.$ $\log 115 = 2.0607$ $\log \sin 4^\circ 12' = \bar{2}.8647$ $\hline .9254$ $\log \sin 20^\circ 41' = \bar{1}.5481$ $\log c_2 = \bar{1}.3773$ $\therefore c_2 = 23.84.$
--	---

$$36. \quad \sin C = \frac{c \sin B}{b} = \frac{1665 \sin 52^\circ 19'}{1325}.$$

$$\log 1665 = 3.2214$$

$$\log \sin 52^\circ 19' = \bar{1}.8984$$

$$\hline 3.1198$$

$$\log 1325 = 3.1222$$

$$\log \sin C = \bar{1}.9976$$

$\therefore C = 84^\circ$ , or  $96^\circ$ , both values being admissible since  $b < c$ .

Now, with diagram of page 132, we have  $\angle BAC_2 = 31^\circ 41'$ ;

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{1325 \sin 31^\circ 41'}{\sin 52^\circ 19'}.$$

$$\log 1325 = 3.1222$$

$$\log \sin 31^\circ 41' = \bar{1}.7203$$

$$\hline 2.8425$$

$$\log \sin 52^\circ 19' = \bar{1}.8984$$

$$\log a = 2.9441; \text{ whence } a = 879.2.$$

$$37. \quad \sin A = \frac{a \sin C}{c} = \frac{324.7 \sin 35^\circ}{421.7}.$$

$$\begin{aligned} \log 324.7 &= 2.5114 \\ \log \sin 35^\circ &= \bar{1}.7586 \\ \hline &2.2700 \\ \log 421.7 &= 2.6250 \\ \bullet \log \sin A &= \bar{1}.6450 \\ \therefore A &= 26^\circ 12', \\ \text{and } B &= 118^\circ 48'. \end{aligned}$$

$$\begin{aligned} b &= \frac{c \sin B}{\sin C} = \frac{421.7 \sin 61^\circ 12'}{35^\circ}, \\ \log 421.7 &= 2.6250 \\ \log \sin 61^\circ 12' &= \bar{1}.9427 \\ \hline &2.5677 \\ \log \sin 35^\circ &= \bar{1}.7586 \\ \log b &= 2.8091 \\ \therefore b &= 644.3. \end{aligned}$$

$$38. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{9.99}{53.91} \cot 17^\circ 30'.$$

$$\begin{aligned} \log 9.99 &= .9996 \\ \log \cot 17^\circ 30' &= .5013 \\ \hline &1.5009 \\ \log 53.91 &= 1.7317 \\ \log \tan \frac{A-B}{2} &= \bar{1}.7692; \end{aligned}$$

$$\begin{aligned} \frac{A+B}{2} &= 72^\circ 30', \\ \text{whence } \frac{A-B}{2} &= 30^\circ 27', \\ \therefore A &= 102^\circ 57', \text{ and } B = 42^\circ 3'. \end{aligned}$$

$$39. \quad \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{549 \times 291}{1000 \times 1258}}.$$

$$\begin{aligned} \log 549 &= 2.7396 \\ \log 291 &= 2.4639 \\ \hline &5.2035 \\ &6.0997 \\ 2) \bar{1}.1038 \end{aligned}$$

$$\begin{aligned} \log 1000 &= 3 \\ \log 1258 &= 3.0997 \\ \hline &6.0997 \end{aligned}$$

$$\log \sin \frac{B}{2} = \bar{1}.5519; \text{ whence } \frac{B}{2} = 20^\circ 52'.$$

$$\therefore B = 41^\circ 44'.$$

$$40. \quad \sin B = \frac{b \sin C}{c} = \frac{17 \sin 43^\circ 12'}{12}.$$

$$\begin{aligned} \log 17 &= 1.2304 \\ \log \sin 43^\circ 12' &= \bar{1}.8354 \\ \hline &1.0658 \\ \log 12 &= 1.0792 \\ \log \sin B &= \bar{1}.9866 \end{aligned}$$

$$\begin{aligned} \therefore B &= 75^\circ 51', \text{ or } 104^\circ 9', \\ \text{both values being admissible} \\ \text{since } c < b. \end{aligned}$$

$$\therefore A = 60^\circ 57', \text{ or } 32^\circ 39'.$$

41.  $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{326 \times 199}{976 \times 451}}$

log 326 =	2.5132		log 976 =	2.9894
log 199 =	2.2989		log 451 =	2.6542
	<u>4.8121</u>			<u>5.6436</u>
	5.6436			
	2) <u>1.1685</u>			

log tan  $\frac{A}{2}$  = 1.5843; whence  $\frac{A}{2} = 21^\circ$   
 $\therefore A = 42^\circ$ .

Again,  $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{451 \times 199}{976 \times 326}}$

log 451 =	2.6542		log 976 =	2.9894
log 199 =	2.2989		log 326 =	2.5132
	<u>4.9531</u>			<u>5.5026</u>
	5.5026			
	2) <u>1.4505</u>			

log tan  $\frac{B}{2}$  = 1.7253; whence  $\frac{B}{2} = 27^\circ 59'$   
 $\therefore B = 55^\circ 58'$ , and  $C = 82^\circ 2'$ .

42.  $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{2.5 \times 1.5}{5 \times 6}} = \sqrt{\frac{1}{8}}$

$\therefore \log \sin \frac{A}{2} = -\frac{3}{2} \log 2 = 1.5485$

$\therefore \frac{A}{2} = 20^\circ 42'$ , and  $A = 41^\circ 24'$ .

43.  $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11.29}{38.95} \cot 23^\circ 37'.5$

log cot  $23^\circ 36' = .3596$   
 subtract for  $1'.5 = \frac{5}{1000}$  (the mean of diffs. 3 and 7)  
.3591

log 11.29 = 1.0527  
1.4118

log 38.95 = 1.5905

log tan  $\frac{B-C}{2} = 1.8213$ ;

$\frac{B+C}{2} = 66^\circ 22'.5$ ,

whence  $\frac{B-C}{2} = 33^\circ 32'$ .

$\therefore B = 99^\circ 54'.5$ , and  $C = 32^\circ 50'.5$ .

$$\text{Again, } a = \frac{b \sin A}{\sin B} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 99^\circ 54' \cdot 5} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 80^\circ 5' \cdot 5}.$$

$$\log 25 \cdot 12 = 1 \cdot 4000$$

$$\log \sin 47^\circ 15' = \bar{1} \cdot 8658$$

$$\hline 1 \cdot 2658$$

$$\log \sin 80^\circ 5' \cdot 5 = \bar{1} \cdot 9935$$

$$\log a = 1 \cdot 2723; \text{ whence } a = 18 \cdot 72.$$

$$44. \quad \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{4367}{4667} \cot 15^\circ 45'.$$

$$\log 4367 = 3 \cdot 6402$$

$$\log \cot 15^\circ 45' = \cdot 5498$$

$$\hline 4 \cdot 1900$$

$$\log 4667 = 3 \cdot 6691$$

$$\log \tan \frac{C-B}{2} = \cdot 5209;$$

$$\text{whence } \frac{C-B}{2} = 73^\circ 14'.$$

$$\therefore C = 147^\circ 29', \text{ and } B = 1^\circ 1'.$$

$$\text{Again, } a = \frac{c \sin A}{\sin C} = \frac{4517 \sin 31^\circ 30'}{\sin 147^\circ 29'} = \frac{4517 \sin 31^\circ 30'}{\sin 32^\circ 31'}.$$

$$\log 4517 = 3 \cdot 6549$$

$$\log \sin 31^\circ 30' = \bar{1} \cdot 7181$$

$$\hline 3 \cdot 3730$$

$$\log \sin 32^\circ 31' = \bar{1} \cdot 7304$$

$$\log a = 3 \cdot 6426; \text{ whence } a = 4391.$$

$$45. \quad \sin C = \frac{c \sin A}{a} = \frac{435 \cdot 6 \sin 36^\circ 18'}{321 \cdot 7}.$$

$$\log 435 \cdot 6 = 2 \cdot 6391$$

$$\log 36^\circ 18' = \bar{1} \cdot 7723$$

$$\hline 2 \cdot 4114$$

$$\log 321 \cdot 7 = 2 \cdot 5074$$

$$\log \sin C = \bar{1} \cdot 9040;$$

whence  $C = 53^\circ 17'$ , or  $126^\circ 43'$ , both values being admissible since  $a < c$ .

46. For the first part of the example, see Art. 197.

$$\tan \theta = \frac{2\sqrt{17 \cdot 32 \times 13 \cdot 47}}{3 \cdot 85} \sin 23^\circ 36'.$$

$$\log 17 \cdot 32 = 1 \cdot 2385$$

$$\log 13 \cdot 47 = 1 \cdot 1294$$

$$2) 2 \cdot 3679$$

$$\hline 1 \cdot 1840$$

$$\log 2 = \cdot 3010$$

$$\log \sin 23^\circ 36' = \bar{1} \cdot 6024$$

$$\hline 1 \cdot 0874;$$

$$1 \cdot 0874$$

$$\log 3 \cdot 85 = \cdot 5855$$

$$\log \tan \theta = \cdot 5019$$

whence  $\theta = 72^\circ 31'$ , approx.

Again, 
$$c = \frac{a-b}{\cos \theta} = \frac{3.85}{\cos 72^\circ 31'}.$$

$$\begin{aligned}\log 3.85 &= .5855 \\ \log \cos 72^\circ 31' &= \bar{1}.4777 \\ \log c &= 1.1078; \text{ whence } c = 12.81.\end{aligned}$$

47. See Art. 195.

$$\tan \phi = \frac{44.1}{10.5} \tan 22^\circ 36'.$$

$$\begin{aligned}\log 44.1 &= 1.6444 \\ \log \tan 22^\circ 36' &= \bar{1}.6194 \\ &\quad \underline{1.2638} \\ \log 10.5 &= 1.0212 \\ \log \tan \phi &= .2426; \text{ whence } \phi = 60^\circ 14', \text{ approx.}\end{aligned}$$

Again, 
$$c = \frac{(a-b) \cos \frac{C}{2}}{\cos \phi} = \frac{10.5 \cos 22^\circ 36'}{\cos 60^\circ 14'}.$$

$$\begin{aligned}\log 10.5 &= 1.0212 \\ \log \cos 22^\circ 36' &= \bar{1}.9653 \\ &\quad \underline{.9865} \\ \log \cos 60^\circ 14' &= \bar{1}.6959 \\ \log c &= 1.2906; \text{ whence } c = 19.53.\end{aligned}$$

### EXAMPLES. XVII. a. PAGE 185.

1. See figure on page 184.

Let  $PC$  represent the cliff, and  $A$  and  $B$  the two objects. Then  $PC = 200$  ft.;  $\angle PAC = 30^\circ$ ,  $\angle PBC = 45^\circ$ .

$$AB = \frac{BP \sin APR}{\sin PAB} = \frac{BP \sin 15^\circ}{\sin 30^\circ};$$

and 
$$BP = \frac{PC}{\sin PBC} = \frac{200}{\sin 45^\circ};$$

$$\therefore AB = \frac{200 \sin 15^\circ}{\sin 45^\circ \sin 30^\circ} = 200(\sqrt{3} - 1) = 146.4 \text{ ft.}$$

2. See figure on page 185.

Let  $P$  represent the mountain top, and  $A, B$  the two positions of the observer.

Then  $\angle PAC = 15^\circ$ ,  $\angle PBC = 75^\circ$ ,  $AB = 1$  mile.

Let  $x$  be the height of mountain in feet;

then  $x = PB \sin 75^\circ$ ; and  $PB = \frac{AB \sin 15^\circ}{\sin 60^\circ}$ ;

$$\therefore x = \frac{1760 \cdot 3 \cdot \sin 15^\circ \sin 75^\circ}{\sin 60^\circ} = \frac{880 \cdot 3 \cdot (\cos 60^\circ - \cos 90^\circ)}{\sin 60^\circ}$$

$$= \frac{880 \cdot 3}{\sqrt{3}} = 880\sqrt{3} = 1524 \text{ ft.}$$

3. Let  $A, B$  be the position of the two forts,  $P$  the first position of the ship, and  $Q$  its position after moving 4 miles towards  $A$ ;

then  $PQ = 4$  miles,  $\angle QPB = 30^\circ$ ,  $\angle AQB = 48^\circ$ ;  $\therefore \angle QBP = 18^\circ$ .

$$\therefore QB = \frac{QP \sin 30^\circ}{\sin 18^\circ} = \frac{8}{\sqrt{5}-1} = 2(\sqrt{5}+1) = 6.472 \text{ miles.}$$

4. See figure on page 184.

Let  $PC$  represent the tower and  $A, B$  the two objects; then

$PC = h$ ,  $\angle PAC = 45^\circ - A$ ,  $\angle PBC = 45^\circ + A$ ,  $\angle APB = 2A$ ;

$$\therefore AB = \frac{PB \sin 2A}{\sin (45^\circ - A)}, \text{ and } PB = \frac{h}{\sin (45^\circ + A)};$$

$$\therefore AB = \frac{2h \sin 2A}{2 \sin (45^\circ - A) \sin (45^\circ + A)} = \frac{2h \sin 2A}{\cos 2A - \cos 90^\circ} = 2h \tan 2A.$$

5. In the figure on page 184, take  $D$  in  $AB$ , so that  $CD = CT$ ; then

$AD = a$  feet,  $DB = b$  feet,  $PC = x$  feet, suppose.

Since

$$\angle CDP = 45^\circ = \angle DPC, \therefore DC = PC = x.$$

Also

$$AC = x + a, BC = x - b, \angle BPC = A.$$

Now from  $\triangle PAC$ ,  $\tan A = \frac{x}{x+a} = \frac{x-b}{x}$ , in  $\triangle BPC$ .

$$\therefore x^2 = (x+a)(x-b); \text{ whence } x = \frac{ab}{a-b}.$$

6. Let  $P, Q$  be the two positions of the observer.

Then from the figure, since

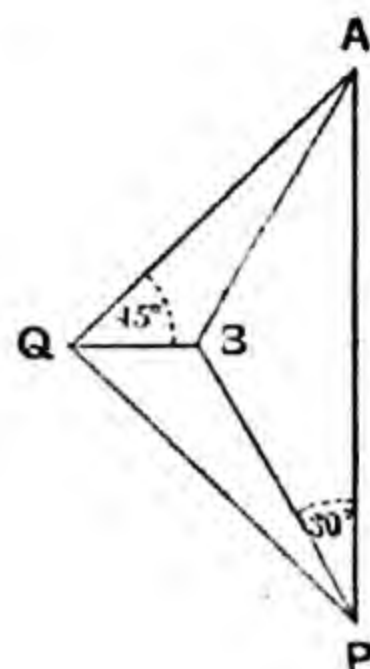
$$\angle AQP = 90^\circ, \quad \angle QPA = 45^\circ;$$

$$\therefore QA = QP = 1 \text{ mile.}$$

And  $\Delta$ 's  $ABQ, PBQ$  are equal in all respects  
(*Euc. I. 4*);

$$\therefore \angle ABQ = \angle PBQ = 120^\circ;$$

$$\therefore AB = \frac{AQ \sin 45^\circ}{\sin 120^\circ} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{3}\sqrt{6} = .816 \text{ miles.}$$



7. Let  $A$  be the base, and  $B$  the top of the tower, and let  $C$  be the point of observation 40 feet up the hill.

Then  $AC = 40$  ft.,  $\angle BAC = 90^\circ - 9^\circ = 81^\circ$ ,  $\angle BCA = 54^\circ$ ;  $\therefore \angle ABC = 45^\circ$ ;

$$\therefore AB = \frac{AC \sin 54^\circ}{\sin 45^\circ} = 10\sqrt{2}(\sqrt{5} + 1) = 45.76 \text{ feet.}$$

8. See figure on page 186.

Let  $PA$  represent the tower, and  $PB$  the flagstaff.

Then  $PB = c$  feet,  $\angle PCA = \alpha$ ,  $\angle PCB = \beta$ , and we have

$$x = CP \sin \alpha = \frac{c \cos (\alpha + \beta) \sin \alpha}{\sin \beta}.$$

9. See figure on page 186.

Let  $BP$  represent the flagstaff, and  $PA$  the wall,  $C$  the point of observation.

Let  $\angle BCP = \alpha$ ,  $\angle PCA = \theta$ ,  $CA = a$ ,

Then  $\tan \alpha = .5$ ,  $BP = 20$  ft.,  $PA = 10$  ft.

$$\text{Now} \quad \tan (\theta + \alpha) = \frac{30}{a}, \quad \tan \theta = \frac{10}{a};$$

$$\therefore 3 \tan \theta = \tan (\theta + \alpha) = \frac{2 \tan \theta + 1}{2 - \tan \theta};$$

$$\therefore 3 \tan^2 \theta - 4 \tan \theta + 1 = 0; \text{ whence } \tan \theta = 1 \text{ or } \frac{1}{3}.$$

10. See figure on page 186.

Let  $BP$  represent the statue,  $PA$  the tower, and  $C$  the point of observation.

Let  $\angle PCA = \alpha$ ,  $\angle BCP = \beta$ ,  $BP = x$  feet.

Then  $PA = 25$  ft.,  $CA = 60$  ft.;

$$\therefore \tan \alpha = \frac{25}{60} = \frac{5}{12}, \text{ and } \tan \beta = \cdot 125 = \frac{1}{8}.$$

Now 
$$x + 25 = 60 \tan (\alpha + \beta) = 60 \left( \frac{\frac{1}{8} + \frac{5}{12}}{1 - \frac{5}{96}} \right) = 34\frac{1}{2};$$

$$\therefore \text{height of statue} = 9\frac{1}{2} \text{ feet.}$$

11. See figure and example on page 187.

Here we have  $BC = 9$  ft.,  $BD = 289$  ft.,  $BE = 324$  ft.;

$$\therefore \tan (\alpha + \theta) = \frac{324}{x}; \quad \tan \alpha = \frac{289}{x}; \quad \tan \theta = \frac{9}{x}.$$

But 
$$\tan (\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta};$$

$$\therefore \frac{324}{x} = \frac{\frac{289}{x} + \frac{9}{x}}{1 - \frac{289}{x} \cdot \frac{9}{x}} = \frac{298}{x} \cdot \frac{x^2}{x^2 - 289 \times 9};$$

$$\therefore 324x^2 - 324 \times 289 \times 9 = 298x^2;$$

$$\therefore 26x^2 = 324 \times 289 \times 9 = 18^2 \times 17^2 \times 3^2;$$

$$\therefore x^2 = \frac{18^2 \times 51^2}{26} = 18^2 \times \left(\frac{50}{5}\right)^2 \text{ nearly};$$

thus  $x = 180$  ft. nearly.

12. See figure on page 187.

Let  $BD$  represent the column,  $DE$  the statue,  $BC$  the man standing by the column,  $A$  the point on the opposite bank of the river.

Then  $BC = 6$  ft.,  $BD = 192$  ft.,  $BE = 216$  ft.

Let  $AB = x$  ft.,  $\angle EAD = \angle CAB = \theta$ ,  $\angle DAB = \alpha$ .

Then 
$$\tan (\alpha + \theta) = \frac{216}{x}; \quad \tan \alpha = \frac{192}{x}; \quad \tan \theta = \frac{6}{x};$$

$$\therefore \frac{216}{x} = \frac{\frac{192}{x} + \frac{6}{x}}{1 - \frac{6}{x} \times \frac{192}{x}} = \frac{198}{x} \cdot \frac{x^2}{x^2 - 6 \times 192}.$$

From this equation we obtain  $x = 48\sqrt{6}$ ;

$\therefore$  breadth of river  $= 48\sqrt{6} = 117.6$  feet nearly.

13. We have at once from a figure,

$$\tan \alpha = \frac{a}{x}, \quad \tan \beta = \frac{b}{x}, \quad \tan \gamma = \frac{c}{x}.$$

Now

$$\alpha + \beta + \gamma = 180^\circ;$$

$$\therefore \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma;$$

$$\therefore \frac{a}{x} + \frac{b}{x} + \frac{c}{x} = \frac{abc}{x^3};$$

that is,

$$(a + b + c) x^2 = abc.$$

14. See figure on page 188.

Let  $P$  be the top of the hill,  $A$  and  $B$  the points of observation;  
then

$$\angle PAC = 45^\circ, \angle BAC = 15^\circ, \angle PDC = 75^\circ;$$

$$\therefore \angle BPA = 75^\circ - 45^\circ = 30^\circ = \angle PAB;$$

$$\therefore PA = 2AB \cos 30^\circ = 500\sqrt{3} \text{ yards};$$

$$\therefore \text{height of hill} = PC = PA \sin 45^\circ = 250\sqrt{6} \text{ yards} \\ = 750\sqrt{6} \text{ feet.}$$

15. We have  $\angle CBA = 30^\circ, \angle BCA = 135^\circ; \therefore \angle BAC = 15^\circ;$

$$\therefore AB = \frac{BC \sin 135^\circ}{\sin 15^\circ} = 1760 \times 3 \times \frac{2}{\sqrt{3}-1} \text{ ft.}$$

$$= 1760 \times 3 (\sqrt{3} + 1) \text{ feet};$$

$$\therefore \text{height of mountain} = AB \sin 60^\circ = 880 \times 3 (3 + \sqrt{3}) = 12492 \text{ ft.}$$

16. See figure on page 188.

Let  $A, B$  be the two points of observation and  $P$  the top of the hill;  
then in the figure  $\angle PAC = \alpha, \angle PDC = \gamma, AB = c$  ft.;

$$\therefore \angle APB = \gamma - \alpha, \angle ABP = \pi - (\gamma - \beta),$$

and

$$AP = \frac{AB \sin (\gamma - \beta)}{\sin (\gamma - \alpha)};$$

$$\therefore \text{height of hill} = AP \sin \alpha$$

$$= c \sin \alpha \sin (\gamma - \beta) \operatorname{cosec} (\gamma - \alpha) \text{ feet.}$$

17. In the figure let  $P$  be the top of the mountain, and  $A, B$  the two points of observation.

Then  $AE = EB = 800$  ft.;

$$\angle BAE = 15^\circ, \angle BED = 30^\circ.$$

Also  $\angle PDC = 75^\circ, \angle PAC = 60^\circ;$

$$\therefore \angle APD = 15^\circ.$$

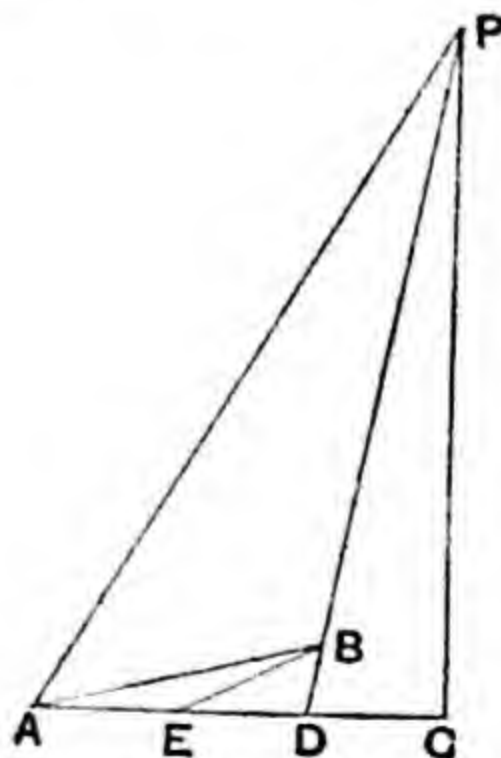
From  $\triangle ABE$  we have

$$AB = 2AE \cos 15^\circ = 1600 \cos 15^\circ \text{ ft.};$$

and from  $\triangle APB, AP = \frac{AB \sin 120^\circ}{\sin 15^\circ}$

$$= 800\sqrt{3} \cot 15^\circ = 800\sqrt{3} (2 + \sqrt{3}) \text{ ft.};$$

$$\therefore \text{height of mountain} = AP \sin 60^\circ \\ = 400 \times 3 (2 + \sqrt{3}) = 4478 \text{ ft.} \\ \text{approximately.}$$



## EXAMPLES. XVII. b. PAGE 190.

1. In the figure of page 186 let  $BP = 25$  ft.,  $PA = 15$  ft.,  $CA = x$  feet

Then  $\frac{BC}{x} = \frac{25}{15} = \frac{5}{3}$  [Euc. vi. 3];

$$\therefore \frac{x^2 + 40^2}{x^2} = \frac{25}{9}; \text{ whence } x = 30;$$

thus the width of the road is 30 ft.

2. Let  $C$  be the position of the observer,  $B$  the top of the statue,  $A$  the foot of the column.

Then if  $CB = x$  feet, we have

$$\frac{x}{CA} = \frac{a}{3a} = \frac{1}{3} \text{ [Euc. vi. 3],}$$

that is,  $\frac{x}{\sqrt{x^2 + 16a^2}} = \frac{1}{3}$ ; whence  $x = a\sqrt{2}$ .

3. See figure on page 189.

Let  $BL$  be the flagstaff,  $LA$  the tower,  $C$  the observer;

then  $BL = a$ ,  $LA = b$ ,  $DA = CE = d$ ,  $EA = CD = h$ ;

also  $BC^2 = CE^2 + EB^2 = d^2 + (a + b - h)^2$ ,

$$CA^2 = CE^2 + EA^2 = d^2 + h^2.$$

Now

$$\frac{BC}{CA} = \frac{BL}{LA}; \quad \frac{(a + b - h)^2 + d^2}{h^2 + d^2} = \frac{a^2}{b^2};$$

$$\therefore \frac{(a + b)^2 - 2h(a + b)}{h^2 + d^2} = \frac{a^2 - b^2}{b^2};$$

or

$$\frac{a + b - 2h}{h^2 + d^2} = \frac{a - b}{b^2};$$

whence

$$(a - b)d^2 = (a + b)b^2 - 2b^2h - (a - b)h^2.$$

4. See figure on page 190.

From  $O$ , the centre of the circle, draw  $OL$ ,  $OM$  perpendicular to  $AB$  and  $DC$  respectively; then  $L$ ,  $M$  bisect  $AB$ ,  $DC$ . Let  $DC = 2x$  feet.

Then  $\angle \beta = \frac{1}{2} \angle DOC \text{ at centre} = \angle COM$ .

Now  $CD = 2x = 2CM = 2OM \tan \beta$   
 $= 2EL \tan \beta = (a + b) \tan \beta$ ,

since

$$2EL = EB + EA = a + b.$$

5. With the same figure and notation as in the last Example, we have  $ED = AB = 20$  ft., and  $\beta = 45^\circ$ .

$$\therefore 2x = (EB + EA) \tan 45^\circ = 2EB + 20;$$

$$\therefore x = EB + 10.$$

Again

$$ED \cdot EC = EB \cdot EA = EB(EB + 20).$$

$$\therefore 20(20 + 2x) = (x - 10)(x + 10);$$

whence

$$x = 50.$$

Thus the height of the column is 100 ft.

6. Take the figure on page 190, interchanging the letters  $A$  and  $B$ . Then we have

$$\angle BDA = \angle BCA = \alpha - \beta; \quad \angle ABD = \angle ACE = 90^\circ - \alpha;$$

$$\therefore \angle DBC = \beta - \angle ABD = \alpha + \beta - 90^\circ.$$

Now from  $\triangle CBD$ ,

$$CD = \frac{BD \sin DBC}{\sin BCE} = \frac{BD \sin (\alpha + \beta - 90^\circ)}{\sin BCE};$$

$$BD = \frac{AB \sin BAD}{\sin BDA} = \frac{a \sin BCE}{\sin (\alpha - \beta)};$$

$$\therefore CD = a \sin (\alpha + \beta - 90^\circ) \operatorname{cosec} (\alpha - \beta).$$

7. See figure on page 191.

Let  $CB$  be the pillar,  $BA$  the pedestal,  $E$  the point where the pillar subtends its maximum angle  $30^\circ$ .

Then using the same construction as in Ex. III. page 191, we have

$$\alpha = 30^\circ, \quad EA = 60 \text{ ft.}$$

$$\text{Now} \quad \angle AEB = \angle ECB = \frac{1}{2} \angle EDB = \frac{1}{2} (90^\circ - 30^\circ) = 30^\circ.$$

$$CB = 2CF = 2DF \tan 30^\circ = 2 \times 60 \times \frac{1}{\sqrt{3}} = 40\sqrt{3} \text{ ft.}$$

8. Let  $O, P$  be the two positions of the observer; let

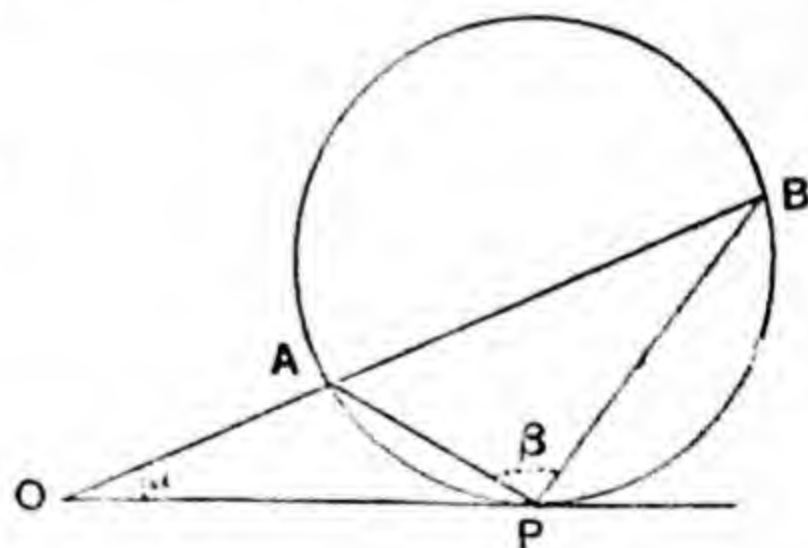
$$\angle APO = \theta.$$

Then  $\angle ABP$ , in alternate segment,  $= \theta$ .

$$\begin{aligned} \text{Now } AB &= \frac{AP \sin \beta}{\sin \theta} \\ &= \frac{c \sin \alpha \sin \beta}{\sin \theta \sin (\alpha + \theta)} \\ &= \frac{2c \sin \alpha \sin \beta}{\cos \alpha - \cos (\alpha + 2\theta)}. \end{aligned}$$

But, from  $\triangle OPB$ ,  $\alpha + 2\theta + \beta = 180^\circ$ ;

$$\therefore AB = 2c \sin \alpha \sin \beta / (\cos \alpha + \cos \beta).$$



9. Let  $OA$  be the tower,  $AB$  the flagstaff,  $P, Q$  the points at which the flagstaff subtends equal angles,  $R$  the point at which it subtends the greatest possible angle; then since  $\angle APB = \angle AQB$ ;

$$\therefore B, A, P, Q \text{ are concyclic and } OP \cdot OQ = OA \cdot OB.$$

Again since  $ARB$  is the greatest angle subtended at a point in  $OQ$  by the str. line  $AB$ ;  $\therefore$  a circle can be drawn to pass through  $A, B$  and touch  $OQ$  at  $R$ ;

$$\therefore OA \cdot OB = OR^2;$$

$$\therefore OP \cdot OQ = OR^2;$$

that is  $OP, OR, OQ$  are in geometrical progression.

10. Here  $ABCD$  is a cyclic quadrilateral;

$$\therefore \frac{AB}{\sin ADB} = \frac{BD}{\sin BAD} = \frac{BD}{\sin BCD} = \frac{DC}{\sin CBD};$$

$$\therefore AB \sin CBD = CD \sin ADB.$$

11. Since  $ABED$  is a cyclic quadrilateral, we have

$$\angle ADC = \angle EBC = \gamma, \text{ and } \angle BDC = \beta.$$

Also  $\angle BDA = \gamma - \beta$ , and  $\angle ACE = \pi - (\alpha + \beta + \gamma)$ .

$$\therefore BC = \frac{BD \sin \beta}{\sin (\alpha + \beta + \gamma)} = \frac{AB \sin (\alpha + \beta) \sin \beta}{\sin (\gamma - \beta) \sin (\alpha + \beta + \gamma)}.$$

12. Here the points  $P, Q, R, S, A$  are concyclic, and

$$\angle RAS = \angle PAQ,$$

since  $AR, AS$  are perpendicular to  $AP, AQ$ ;

$$\therefore PQ = RS = \sqrt{400 + 100 - 2 \times 200 \cos 30^\circ} \\ = \sqrt{500 - 200\sqrt{3}} = 12.4 \text{ ft. nearly.}$$

13. Let  $A, B$  be the two beacons,  $P, Q$  the positions of the ship at the end of 3 min. and 21 min. respectively. Let  $\angle ABP = \alpha$ ,  $\angle PBQ = \theta$ . Then it is easily seen that

$$\angle OAP = 90^\circ + \alpha,$$

$$\angle OAQ = 90^\circ + \alpha + \theta.$$

Also from the  $\triangle OBQ$  we have

$$\alpha + \theta + 90^\circ + \alpha = 135^\circ,$$

so that

$$\alpha + \theta = 45^\circ - \alpha.$$

If

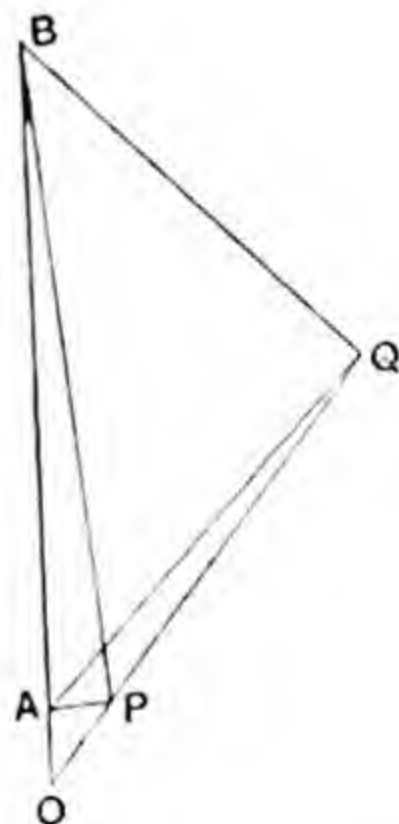
$$PO = x, OQ = 7x;$$

hence

$$x = \frac{AP}{\sin 45^\circ} \sin (90^\circ + \alpha) \\ = AB \sqrt{2} \cdot \sin \alpha \cos \alpha \\ = \frac{5}{\sqrt{2}} \sin 2\alpha \dots\dots\dots(1).$$

Again

$$7x = \frac{AQ}{\sin 45^\circ} \sin (90^\circ + \alpha + \theta) \\ = AB \sqrt{2} \sin (\alpha + \theta) \cos (\alpha + \theta) \\ = \frac{5}{\sqrt{2}} \sin 2(\alpha + \theta) = \frac{5}{\sqrt{2}} \sin (90 - 2\alpha) = \frac{5}{\sqrt{2}} \cos 2\alpha \dots\dots\dots(2).$$



Squaring and adding (1) and (2), we have

$$50x^2 = \frac{25}{2}; \therefore x = \frac{1}{2} \text{ mile.}$$

$\therefore$  the ship sails a mile in 6 min., or at the rate of 10 miles an hour.

Again if  $y$  miles be the distance from  $O$  at which the beacons subtend the greatest angle, we have

$$y^2 = OP \cdot OQ = \frac{7}{4}; \therefore y = \frac{\sqrt{7}}{2}.$$

And the ship will travel this distance in  $\frac{\sqrt{7}}{2} \times \frac{60}{10}$ , or  $3\sqrt{7}$  minutes.

14. Let  $AB$  be the flag-staff,  $BC$  the tower,  $D$  and  $E$  the first and second positions of the observer respectively.

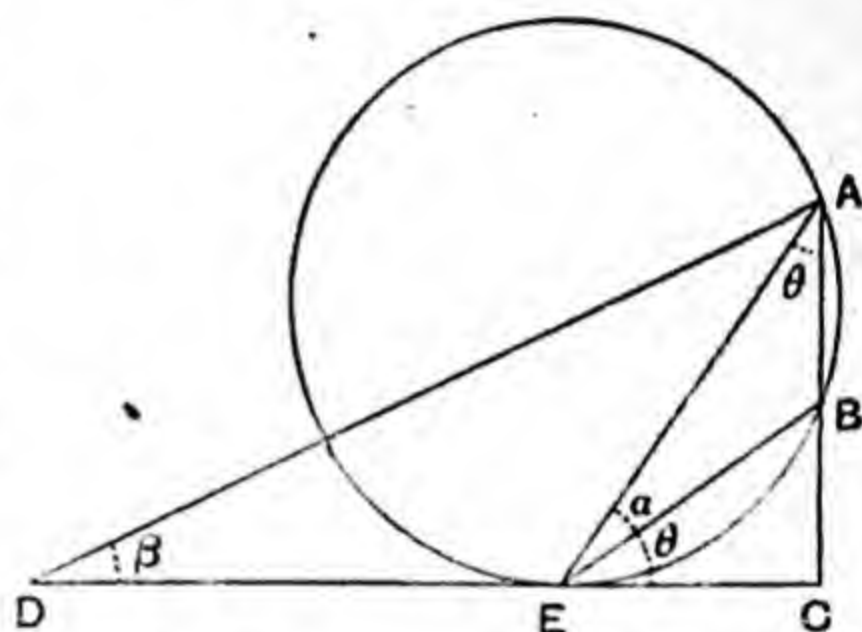
Then since  $AB$  subtends the maximum angle at  $E$ , the circle round  $ABE$  touches  $DC$  at  $E$ , so that

$\angle BEC = \angle EAB = \theta$ , suppose.

$\therefore \alpha + \theta = \angle EBC = 90^\circ - \theta$ ;

$\therefore 2\theta = 90^\circ - \alpha$ ;

also  $\angle EAD = \alpha + \theta - \beta$   
 $= 90^\circ - \theta - \beta$ .



Now  $AB = \frac{AE \sin \alpha}{\sin (\theta + \alpha)}$ ,  $AE = \frac{ED \sin \beta}{\sin (90^\circ - \theta - \beta)} = \frac{a \sin \beta}{\cos (\theta + \beta)}$ ;

$$\therefore AB = \frac{a \sin \alpha \sin \beta}{\sin (\theta + \alpha) \cos (\theta + \beta)} = \frac{2a \sin \alpha \sin \beta}{\sin (2\theta + \alpha + \beta) + \sin (\alpha - \beta)}$$

$$= \frac{2a \sin \alpha \sin \beta}{\cos \beta + \sin (\alpha - \beta)}.$$

### EXAMPLES. XVII. c. PAGE 195.

1. Let  $A$  be the top of the hill and  $B$  its projection on the horizontal plane through  $P, Q$ .

Let  
 then

$$AB = x \text{ yards;}$$

$$BP = BA = x,$$

$$BQ = BA \cot 30^\circ = x\sqrt{3};$$

$$\therefore 3x^2 = x^2 + 500^2;$$

$$\therefore x = 250\sqrt{2};$$

that is, height of the hill  $= 250\sqrt{2}$  yards  $= 1060.5$  feet.

2. Let  $P$  be the top, and  $Q$  the bottom of the spire;

then

$$AQ = 250 \cot 60^\circ = \frac{250}{\sqrt{3}} \text{ feet,}$$

$$BQ = 250 \cot 30^\circ = 250\sqrt{3} \text{ feet;}$$

$$\therefore AB^2 = BQ^2 - AQ^2 = 250^2 \left( 3 - \frac{1}{3} \right);$$

$$\therefore AB = 250 \cdot \frac{2\sqrt{2}}{\sqrt{3}} = \frac{500\sqrt{6}}{3} \text{ feet.}$$

3. Let  $P$  be the top and  $Q$  the bottom of the tower;

then  $\angle PAQ = 60^\circ$ ;  $\therefore QA = 360 \cot 60^\circ = \frac{360}{\sqrt{3}}$  feet and  $QB = QP = 360$  feet;

$$\therefore \text{breadth of river} = AB = \sqrt{BQ^2 - QA^2} = 360 \sqrt{1 - \frac{1}{3}} = 120\sqrt{6} \text{ feet.}$$

4. See figure on page 194.

Let  $CD$  be the steeple, then  $\angle CAD = 45^\circ$ ;  $\therefore CA = CD = x$ ;

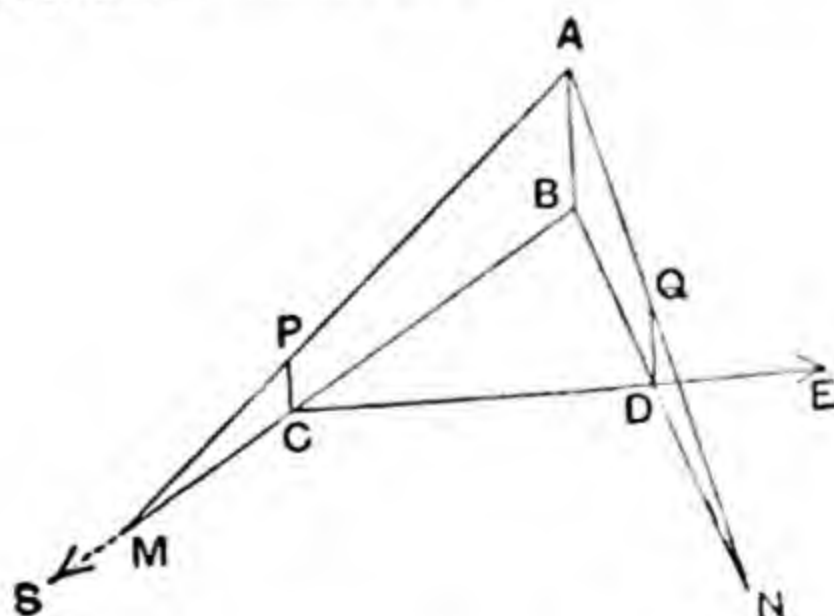
and  $\angle CBD = 15^\circ$ ;  $\therefore CB = x \cot 15^\circ = x(2 + \sqrt{3})$ ;

$$\therefore x^2(7 + 4\sqrt{3}) = x^2 + 4a^2;$$

$$\therefore x^2 = \frac{4a^2}{2\sqrt{3} + 3} = \frac{a^2(4 - 2\sqrt{3})}{\sqrt{3}};$$

$$\therefore \text{height of steeple} = \frac{a(\sqrt{3} - 1)}{\sqrt{3}} = a(3^{\frac{1}{4}} - 3^{-\frac{1}{4}}).$$

5. Let  $AB$  be the lighthouse, and let  $CP, DQ$  represent the two positions of the observer,  $M, N$  the extremities of his shadow at each place.



Let  $AB = x$  feet, then  $x : BM = PC : CM = 6 : 24$ ;

$$\therefore BC = BM - CM = 4x - 24.$$

Also

$$\begin{aligned}x : BN &= QD : DN = 6 : 30; \\ \therefore BD &= BN - DN = 5x - 30; \\ \therefore 25(x - 6)^2 &= 300^2 + 16(x - 6)^2; \\ \therefore 9(x - 6)^2 &= 300^2;\end{aligned}$$

whence

$$x = 106 \text{ or } -94;$$

$\therefore$  the light is 106 feet from the ground.

6. Let  $A$  be the balloon, and  $C, B, D$  the points of observation at which the angles of elevation of the balloon are  $60^\circ, 30^\circ, 45^\circ$  respectively.

Let  $x$  yards be the height of the balloon;

$$\text{then } AB = x \operatorname{cosec} 30^\circ = 2x, \quad AC = x \operatorname{cosec} 60^\circ = \frac{2x}{\sqrt{3}}, \quad AD = x \operatorname{cosec} 45^\circ = x\sqrt{2}.$$

Now

$$2AD^2 + 2BD^2 = AB^2 + AC^2;$$

$$\therefore 4x^2 + 2 \times 880^2 = 4x^2 + \frac{4x^2}{3};$$

whence

$$x = 880 \cdot \sqrt{\frac{3}{2}} = 440\sqrt{6};$$

$$\therefore \text{height of balloon} = 440\sqrt{6} \text{ yards.}$$

7. Let  $A$  be the top of the mountain,  $BC$  the base of length  $2a$ ,  $D$  its middle point.

Let  $x$  be the height of the mountain;

then

$$AB = AC = x \operatorname{cosec} \theta,$$

$$AD = x \operatorname{cosec} \phi.$$

And we have

$$AD^2 + BD^2 = AB^2;$$

$$\therefore x^2 \operatorname{cosec}^2 \phi + a^2 = x^2 \operatorname{cosec}^2 \theta;$$

$$\therefore x^2 = \frac{a^2}{\operatorname{cosec}^2 \theta - \operatorname{cosec}^2 \phi} = \frac{a^2 \sin^2 \theta \cos^2 \theta}{\sin^2 \phi - \sin^2 \theta},$$

$$\therefore x = \frac{a \sin \theta \cos \theta}{\sqrt{\sin(\phi + \theta) \sin(\phi - \theta)}};$$

$$\therefore \text{height of mountain is } a \sin \theta \cos \theta \sqrt{\operatorname{cosec}(\phi + \theta) \operatorname{cosec}(\phi - \theta)}.$$

8. Let  $AB, CD$  be the two vertical poles,  $E$  the point in the line  $BD$  joining their feet at which each subtends an angle  $\alpha$ , and  $F$  any point in the horizontal plane such that  $\angle DFB$  is a right angle;

then

$$\angle CED = \angle AEB = \alpha, \quad \angle AFB = \beta, \quad \angle CED = \gamma;$$

and

$$EB = AB \cot AEB = a \cot \alpha, \quad ED = b \cot \alpha;$$

$$\therefore BD = (a + b) \cot \alpha;$$

$$BF = AB \cot AFB = a \cot \beta, \quad DF = b \cot \gamma;$$

and

$$BD^2 = BF^2 + DF^2;$$

$$\therefore (a + b)^2 \cot^2 \alpha = a^2 \cot^2 \beta + b^2 \cot^2 \gamma.$$

9. Let  $A$  be the top of the hill,  $B$  its projection on the horizontal plane in which the road lies,  $C, D, E$  the three consecutive milestones whose angles of depression are observed.

Then  $\angle ACB = \alpha$ ,  $\angle ADB = \beta$ ,  $\angle AEB = \gamma$ ,  
and  $ED = DC = 1760 \times 3$  feet.

Let  $x$  feet be the height of the hill;

then since  $BC^2 + BE^2 = 2BD^2 + 2DE^2$ ;  
 $\therefore x^2 \cot^2 \alpha + x^2 \cot^2 \gamma = 2x^2 \cot^2 \beta + 2 \times 1760^2 \times 3^2$ ;  
 $\therefore x = \frac{520\sqrt{2}}{\sqrt{\cot^2 \alpha - 2 \cot^2 \beta + \cot^2 \gamma}}$ .

10. Let  $P, Q$  be the two positions of the observer.  
Let  $x$  feet be the height of the chimneys;

then  $AP = x \cot 60^\circ = \frac{x}{\sqrt{3}}$ ,  $AQ = AB = x$ ;  
 $\therefore x^2 = \frac{x^2}{3} + 80^2$ ;

whence  $x = 40\sqrt{6}$ ;  $\therefore$  height of chimney  $= 40\sqrt{6}$  feet.

Again  $CQ = x \cot 30^\circ = x\sqrt{3}$ ;

$\therefore CP = \sqrt{CQ^2 - PQ^2} = \sqrt{3 \times 40^2 \times 6 - 80^2} = 40\sqrt{14}$ ;  
 $\therefore AC = 40\sqrt{14} + 40\sqrt{2} = 40(\sqrt{14} + \sqrt{2}) = 206$  feet nearly.

11. Let  $A$  be the balloon,  $C, D$  the positions of the two observers,  $B$  the point vertically below  $A$  in the horizontal plane of the observers.

Let  $AB = x$  yds.  $=$  height of the balloon;  
then  $CD = 500$  yards,

$BC = x \cot 60^\circ = \frac{x}{\sqrt{3}}$ ,  $BD = x$ ;

and  $CD^2 = BC^2 + BD^2 - 2BC \cdot BD \cos 45^\circ$ ;

$\therefore 500^2 = \frac{x^2}{3} + x^2 - \frac{x^2\sqrt{6}}{3}$ ;

whence  $x^2 = 50^2(120 + 30\sqrt{6})$ ;

$\therefore$  height of balloon  $= 50\sqrt{120 + 30\sqrt{6}} = 696$  yds. nearly.

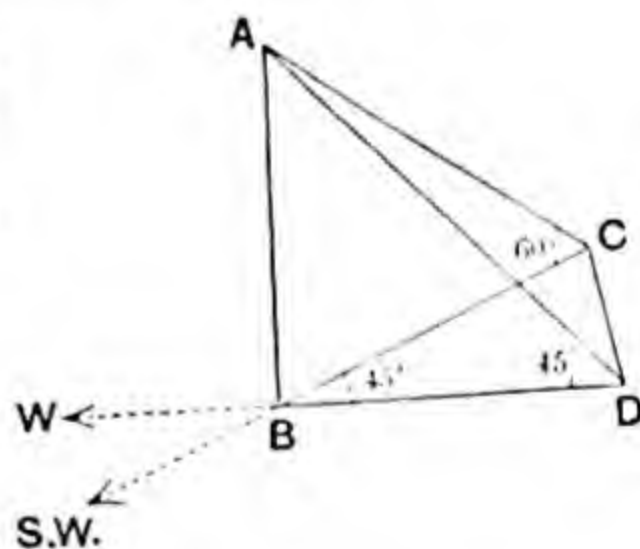
12. In the diagram on page 195, let  $AC$  be a line of greatest slope on the hill, and let  $CH$  be the railway.

Then  $\angle ACB = \alpha$ ,  $\angle BCG = x$ ,  $\angle HCG = \beta$ , and  $\angle CHG$  is a right angle.  
Let  $AB = HG = h$ .

Then

$BC = h \cot \alpha$ ,  $CG = h \cot \beta$ .

$\therefore \cos x = \frac{BC}{CG} = \cot \alpha \tan \beta$ .



## EXAMPLES. XVII. d. PAGE 197.

1. Let  $C_1, C_2$  be the positions of the two churches,  $M$  the position of the man. Then  $MC_1C_2$  is an isosceles triangle, and the vertical angle at  $M = 11^\circ 24'$ .

If  $x$  miles is the height of the balloon, we have

$$\begin{aligned} x &= \frac{1}{2} \cot 5^\circ 42' \\ &= \frac{1}{2} \times 10.02 \\ &= 5.01. \end{aligned}$$

2. Let  $DC$  be the pole, then from a figure we have

$$DC = BC \sin 10^\circ 45' = \frac{100 \sin 5^\circ 30' \times \sin 10^\circ 45'}{\sin 5^\circ 15'}.$$

$$\log 100 + \log \sin 5^\circ 30' = .9816$$

$$\log \sin 10^\circ 45' = \bar{1}.2707$$

---


$$.2523$$

$$\log \sin 5^\circ 15' = \bar{2}.9612$$

---


$$\log DC = 1.2911$$

$$\therefore DC = 19.54 \text{ yards.}$$

Again,

$$BC = \frac{DC}{\tan 10^\circ 45'}$$

$$\log DC = 1.2911$$

$$\log \tan 10^\circ 45' = \bar{1}.2785$$

---


$$\log BC = 2.0126$$

$$\therefore BC = 102.9 \text{ yards.}$$

3. If the required height is  $x$  feet, we have

$$x = \frac{500 \sin 26^\circ 34' \times \sin 12^\circ 32'}{\sin 14^\circ 2'}$$

$$\log 500 = 2.6990$$

$$\log \sin 26^\circ 34' = 1.6505$$

$$\log \sin 12^\circ 32' = 1.3364$$

$$\hline 1.6859$$

$$\log \sin 14^\circ 2' = 1.3847$$

$$\log x = 2.3012$$

$$\therefore x = 200.1.$$

4. Let  $B$  be the position of the boat,  $FC$  the flagstaff,  $CD$  the cliff, then, if  $CD = x$  feet, we have

$$x = \frac{30 \sin 43^\circ 46'}{\sin 2^\circ 6'} \times \sin 44^\circ 8'.$$

$$\log 30 = 1.4771$$

$$\log \sin 43^\circ 46' = 1.8399$$

$$\log \sin 44^\circ 8' = 1.8429$$

$$\hline 1.1599$$

$$\log \sin 2^\circ 6' = 2.5640$$

$$\log x = 2.5959$$

$$\therefore \text{height} = 394.4 \text{ ft.}$$

Again,

$$BD = \frac{x}{\tan 44^\circ 8'}$$

$$\log x = 2.5959$$

$$\log \tan 44^\circ 8' = 1.9869$$

$$\log BD = 2.6090$$

$$\therefore \text{distance} = 406.4 \text{ ft.}$$

5. With the figure on page 184, we have

$$\angle PBC = 10^\circ, \angle PAC = 5^\circ; PB = AB = 5280 \text{ ft.}$$

Now

$$PC = PB \sin 10^\circ = 5280 \sin 10^\circ.$$

$$\log 5280 = 3.7226$$

$$\log \sin 10^\circ = 1.2397$$

$$\hline 2.9623$$

$$\therefore PC = 916.8 \text{ ft.}$$

Again,

$$BC = BP \cos 10^\circ$$

$$= (1 \times \cos 10^\circ) \text{ miles}$$

$$= .9848 \text{ miles.}$$

6. Let  $B$  be the point in the road which is vertically below the observer  $A$ . Let  $DC$  be the telegraph post, and let a horizontal line through  $A$  meet  $DC$  in  $E$ . Then  $\angle DAE = 17^\circ 19'$ ,  $\angle ACB = 8^\circ 36'$ .

$$AE = BC = \frac{15}{\tan 8^\circ 36'}.$$

$$\log 15 = 1.1761$$

$$\log \tan 8^\circ 36' = \bar{1}.1797$$

$$\log AE = 1.9964$$

$$\therefore AE = 99.17 \text{ ft.}$$

Again,

$$DE = EA \tan 17^\circ 19'.$$

$$\log EA = 1.9964$$

$$\log \tan 17^\circ 19' = \bar{1}.4938$$

$$\log DE = 1.4902$$

$$\therefore DE = 30.91 \text{ ft.,}$$

$$\text{and } DC = 45.91 \text{ ft.}$$

7. Here we may take the third figure on page 131. Then  $AC = 60$  miles,  $CB_2 = CB_1 = 30$  miles,  $\angle CAB_2 = 20^\circ 16'$ .

$$\sin B = \frac{60}{30} \sin 20^\circ 16' = 2 \times .3464 = .6928;$$

$$\therefore B = 43^\circ 51', \text{ or } 136^\circ 9' \text{ (since } a < b \text{)}.$$

$$\therefore \angle ACB_2 = 23^\circ 35', \quad \angle ACB_1 = 180^\circ - 64^\circ 7'.$$

Now

$$AB_2 = \frac{30 \sin 23^\circ 35'}{\sin 20^\circ 16'}.$$

$$\log 30 = 1.4771$$

$$\log \sin 23^\circ 35' = \bar{1}.6022$$

$$1.0793$$

$$\log \sin 20^\circ 16' = \bar{1}.5396$$

$$\log AB_2 = 1.5397$$

$$\therefore AB_2 = 34.65 \text{ miles.}$$

Again,

$$AB_1 = \frac{30 \sin 64^\circ 7'}{\sin 20^\circ 16'}$$

$$\log 30 = 1.4771$$

$$\log \sin 64^\circ 7' = \bar{1}.9541$$

$$\hline 1.4312$$

$$\log \sin 20^\circ 16' = \bar{1}.5396$$

$$\log AB_1 = 1.8916$$

$$\therefore AB_1 = 77.91 \text{ miles.}$$

Thus the train must travel at the rate of 11.55 miles or 25.97 miles per hour.

8. Let  $C$  be the doorstep,  $E$  the point of observation on the roof; then  $CE = h$ . Let  $AB$  be the spire and let  $ED$  drawn horizontally meet  $AB$  in  $D$ ; then  $\angle ACB = 5a$ ,  $\angle AED = 4a$ . Also  $\angle EAC = a$ . Let  $AB = x$ .

$$\text{Then } x = AC \sin 5a = \frac{h \sin (90^\circ + 4a)}{\sin a} \cdot \sin 5a$$

$$= h \operatorname{cosec} a \cos 4a \sin 5a.$$

$$CB = x \cot 5a = h \operatorname{cosec} a \cos 4a \cos 5a.$$

In the particular case

$$a = 7^\circ 19', \quad 4a = 29^\circ 16', \quad 5a = 36^\circ 35'.$$

$$\text{Then } x = \frac{39 \cos 29^\circ 16' \times \sin 36^\circ 35'}{\sin 7^\circ 19'}.$$

$$\log 39 = 1.5911$$

$$\log \cos 29^\circ 16' = \bar{1}.9407$$

$$\log \sin 36^\circ 35' = \bar{1}.7753$$

$$\hline 1.3071$$

$$\log \sin 7^\circ 19' = \bar{1}.1050$$

$$\log x = 2.2021$$

$$\therefore x = 159.2; \text{ thus the height is } 159.2 \text{ ft.}$$

Again,

$$CB = \frac{x}{\tan 5a} = \frac{x}{\tan 36^\circ 35'}$$

$$\log x = 2.2021$$

$$\log \tan 36^\circ 35' = \bar{1}.8705$$

$$\log CB = 2.3316$$

Thus the distance is 214.5 ft.

**EXAMPLES. XVIII. a. PAGE 206.**

1.  $\text{Area} = \frac{1}{2} \times 300 \times 120 \sin 150^\circ = 300 \times 60 \times \frac{1}{2} = 9000 \text{ sq. feet.}$

2.  $2s = 171 + 204 + 195 = 570;$

$$\therefore \text{area} = \sqrt{285 \times 114 \times 81 \times 90} = \sqrt{10^2 \times 57^2 \times 27^2} = 15390.$$

3. Let  $a = 70$ ,  $b = 147$ ,  $c = 119$ ; then  $s = 168$ .

$$\therefore \sin B = \frac{2}{70 \times 119} \sqrt{168 \times 98 \times 21 \times 49} = \frac{2 \times 12 \times 7 \times 49}{70 \times 119} = \frac{84}{85}.$$

4. Let  $a = 39$ ,  $b = 40$ ,  $c = 25$ , and denote the perpendiculars by  $p_1$ ,  $p_2$ ,  $p_3$ .

Then  $\text{area} = \sqrt{52 \times 13 \times 12 \times 27} = 12 \times 13 \times 3.$

$$\therefore p_1 = \frac{2\Delta}{a} = 24, \quad p_2 = \frac{2\Delta}{b} = \frac{117}{5}, \quad p_3 = \frac{2\Delta}{c} = \frac{936}{25}.$$

5.  $\text{Area} = \frac{30^2 \sin 22\frac{1}{2}^\circ \sin 112\frac{1}{2}^\circ}{2 \sin 135^\circ} = \frac{30^2 \sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ}{2 \sin 45^\circ} = \frac{30^2}{4} = 225 \text{ sq. ft.}$

6. The diagonal bisects the parallelogram;

$$\therefore \text{area} = 42 \times 32 \sin 30^\circ = 672 \text{ sq. feet.}$$

7. Let  $a$  yds. be the length of a side.

Then  $\text{area} = a^2 \sin 150^\circ = \frac{a^2}{2};$

$$\therefore \frac{a^2}{2} = 648; \therefore a = 36 \text{ yards.}$$

$$\therefore \text{length of a side is 36 yds.}$$

8.  $13 + 14 + 15 = 42;$

$$\therefore \Delta = \sqrt{21 \times 8 \times 7 \times 6} = 4 \times 3 \times 7.$$

$$\therefore R = \frac{13 \times 14 \times 15}{4\Delta} = \frac{65}{8} = 8\frac{1}{8}.$$

$$r = \frac{\Delta}{21} = 4.$$

9.  $17 + 10 + 21 = 48;$

$$\therefore \Delta = \sqrt{24 \times 7 \times 14 \times 3} = 4 \times 3 \times 7.$$

$$\therefore r_1 = \frac{\Delta}{7} = 12, \quad r_2 = \frac{\Delta}{14} = 6, \quad r_3 = \frac{\Delta}{3} = 28.$$

10.  $s - a = \frac{\Delta}{r_1} = 12; s - b = \frac{\Delta}{r_2} = 8; s - c = \frac{\Delta}{r_3} = 4; \therefore s = 24;$

$$\therefore a = 12, b = 16, c = 20.$$

$$11. \sqrt{rr_1r_2r_3} = \sqrt{\frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}} = \Delta.$$

$$12. s(s-a) \tan \frac{A}{2} = s(s-a) \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{s(s-a)(s-b)(s-c)} = \Delta$$

$$13. rr_1 \cot \frac{A}{2} = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{s(s-a)}{\Delta} = \Delta.$$

$$14. 4Rs = \frac{abc}{\Delta} \cdot \frac{\Delta}{s} \cdot s = abc.$$

$$15. r_1r_2r_3 = \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{\Delta^3 s}{\Delta^2} = \Delta s = rs^2.$$

$$16. r \cot \frac{B}{2} \cot \frac{C}{2} = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \cdot \cot \frac{B}{2} \cot \frac{C}{2} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} = r_1.$$

$$17. \text{First side} = r \left( \frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) = rs = \Delta.$$

$$18. r_1r_2 + rr_3 = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{s(s-c)} = \frac{\Delta^2 \{ s(s-c) + (s-a)(s-b) \}}{\Delta^2} \\ = 2s^2 - (a+b+c)s + ab = ab.$$

$$20. r_1 + r_2 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \\ = 4R \cos \frac{C}{2} \cdot \sin \frac{A+B}{2} = 4R \cos^2 \frac{C}{2} \\ = \frac{2c}{\sin C} \cdot \cos^2 \frac{C}{2} = c \cot \frac{C}{2}.$$

$$21. \text{As in Ex. 20, } r_1 - r = 4R \sin \frac{A}{2} \cdot \cos \frac{B+C}{2} = 4R \sin^2 \frac{A}{2}$$

$$r_2 + r_3 = 4R \cos^2 \frac{A}{2}.$$

$$\therefore (r_1 - r)(r_2 + r_3) = 4R^2 \sin^2 A = a^2.$$

22. By the formulæ of Art. 212, we have

$$\frac{r_1}{r} = \cot \frac{B}{2} \cot \frac{C}{2}; \therefore r_1 \cot \frac{A}{2} = r \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$23. \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a+s-b+s-c}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

$$24. \quad r_2 r_3 + r_3 r_1 + r_1 r_2 = \frac{\Delta^2}{(s-b)(s-c)} + \dots + \dots = s(s-a) + \dots + \dots \\ = 3s^2 - (a+b+c)s = s^2.$$

$$25. \quad \text{As in Ex. 21, } r_2 + r_3 = 4R \cos^2 \frac{A}{2}, \quad r_1 - r = 4R \sin^2 \frac{A}{2}.$$

$$\therefore r_1 + r_2 + r_3 - r = 4R \left( \cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} \right) = 4R.$$

$$26. \quad r_1 + r_2 = 4R \cos^2 \frac{C}{2}, \quad r_3 - r = 4R \sin^2 \frac{C}{2}. \quad [\text{Ex. 21.}]$$

$$\therefore r + r_1 + r_2 - r_3 = 4R \left( \cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) = 4R \cos C.$$

$$27. \quad b^2 \sin 2C + c^2 \sin 2B = 2b \sin C \cdot b \cos C + 2c \sin B \cdot c \cos B \\ = 2b \sin C (b \cos C + c \cos B) = 2ab \sin C = 4\Delta.$$

$$28. \quad (a+b) \sec \frac{A-B}{2} = 2R (\sin A + \sin B) \sec \frac{A-B}{2} \\ = 4R \sin \frac{A+B}{2} \cos \frac{A-B}{2} \sec \frac{A-B}{2} = 4R \cos \frac{C}{2}.$$

$$29. \quad a^2 - b^2 = 4R^2 (\sin^2 A - \sin^2 B) = 4R^2 \sin (A+B) \sin (A-B) \\ = 4R^2 \sin C \sin (A-B) = 2Rc \sin (A-B).$$

$$30. \quad \text{First side} = 4R^2 \cdot \frac{\sin^2 A - \sin^2 B}{2} \cdot \frac{\sin A \sin B}{\sin (A-B)} \\ = \frac{4R^2 \sin (A+B) \sin A \sin B}{2} \\ = \frac{2R \sin A \cdot 2R \sin B \cdot \sin C}{2} = \frac{ab}{2} \sin C = \Delta.$$

$$31. \quad (1) \quad \text{We have } 2\Delta = ap_1 = bp_2 = cp_3;$$

$$\therefore \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a+b+c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

$$(2) \quad \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a+b-c}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}.$$

$$32. \quad (r_1 - r)(r_2 - r)(r_3 - r) = 64R^3 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \quad [\text{Ex. 21}] \\ = 4Rr^2. \quad [\text{Art. 212.}]$$

$$33. \left(\frac{1}{r} - \frac{1}{r_1}\right) \left(\frac{1}{r} - \frac{1}{r_2}\right) \left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{(r_1 - r)(r_2 - r)(r_3 - r)}{r^2 r_1 r_2 r_3} = \frac{4Rr^2}{r^2 \Delta^2} \quad [\text{Ex. 11}]$$

$$= \frac{4R}{r^2 s^2}.$$

$$34. 4\Delta (\cot A + \cot B + \cot C) = 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$= b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$$

$$= a^2 + b^2 + c^2.$$

$$35. \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = \frac{(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)}{\Delta}$$

$$= \frac{s(b-c+c-a+a-b) - \frac{1}{2}a(b-c) + b(c-a) + c(a-b)}{\Delta} = 0$$

$$36. a^2 b^2 c^2 (\sin 2A + \sin 2B + \sin 2C) = 4a^2 b^2 c^2 \sin A \sin B \sin C$$

$$= 4 \cdot bc \sin A \cdot ca \sin B \cdot ab \sin C$$

$$= 32\Delta^2.$$

$$37. a \cos A + b \cos B + c \cos C = 2R (\sin A \cos A + \sin B \cos B + \sin C \cos C)$$

$$= R (\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C.$$

$$38. a \cot A + b \cot B + c \cot C = 2R (\cos A + \cos B + \cos C)$$

$$= 2R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)$$

$$= 2(R+r).$$

$$39. (b+c) \tan \frac{A}{2} = 2R (\sin B + \sin C) \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = 4R \cos \frac{B-C}{2} \cos \frac{B+C}{2}$$

$$= 2R (\cos B + \cos C);$$

$$\therefore (b+c) \tan \frac{A}{2} + \text{two similar terms} = 4R (\cos A + \cos B + \cos C).$$

$$40. r (\sin A + \sin B + \sin C) = 4r \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 16R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 2R \sin A \sin B \sin C.$$

$$\begin{aligned}
 41. \quad \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} &= \frac{1}{2} (3 + \cos A + \cos B + \cos C) \\
 &= \frac{1}{2} \left( 4 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \\
 &= 2 + \frac{r}{2R}.
 \end{aligned}$$

**EXAMPLES. XVIII. b. PAGE 210.**

1. By Art. 214, the area  $= \frac{1}{2} \cdot 10 \cdot 9 \cdot \sin 36^\circ$   
 $= 45 \times .588 = 26.46 \text{ sq. ft.}$

2. By Art. 215, the perimeter  $= 2 \cdot 15 \cdot \frac{3}{2} \cdot \tan 12^\circ$   
 $= 45 \times .213 = 9.585 \text{ yds.}$

The area  $= 15 \cdot \frac{9}{4} \cdot \tan 12^\circ = \frac{135 \times .213}{4} = 7.18875 \text{ sq. yds.}$

3. In fig. of Art. 215 let  $AB = 2x =$  side of regular hexagon. Then  
 $OD = x \cot 30^\circ = x\sqrt{3}.$   
 $\therefore$  area of inscribed circle  $= 3\pi x^2.$

Again, in fig. of Art. 214, if  $AB$  be the side of the hexagon,  
 $OA = AD \operatorname{cosec} 30^\circ = 2x.$   
 $\therefore$  area of circumscribed circle  $= 4\pi x^2.$   
 $\therefore$  ratio of areas is 3 to 4.

4. With fig. and notation of Art. 217,  
 area of pentagon  $= 5AD \cdot OD = 5r^2 \tan 36^\circ;$   
 $\therefore 250 = 5r^2 \tan 36^\circ; \pi r^2 = \frac{22}{7} \times 50 \cot 36^\circ = 216.23 \text{ sq. ft.}$

5. Area of circle  $= \pi r^2 = 1386$ ; whence  $r = 21.$

By Art. 214, perimeter  $= 16r \sin 22^\circ 30' = 16 \times 21 \times .382 = 128.352 \text{ in.}$

6. Area of circle  $= \pi r^2 = 616$ ; whence  $r = 14.$

By Art. 215, perimeter of pentagon  $= 2 \cdot 5 \cdot 14 \tan 36^\circ = 140 \times .727$   
 $= 101.78 \text{ ft.}$

7. Area of circle  $= \pi r^2 = 2464$ ; whence  $r = 28.$

Now in Art. 214, if  $AB$  is a side of the quindecagon,  $OD = 28$ , and  
 diameter of required circle  $= 2AO = 2OD \sec 12^\circ = 2 \times 28 \times 1.022 = 57.232 \text{ ft.}$

8. Let  $r$  be the radius of the circle; then

$$\text{for the pentagon,} \quad 50 = \frac{5}{2} r^2 \sin \frac{2\pi}{5},$$

$$\text{for the dodecagon,} \quad \text{area} = \frac{12}{2} r^2 \sin \frac{\pi}{6}; \quad [\text{Art. 214}]$$

$$\therefore \frac{\text{area of dodecagon}}{50} = \frac{12}{5} \cdot \frac{\sin 30^\circ}{\sin 72^\circ};$$

$$\therefore \text{area of dodecagon} = 60 \operatorname{cosec} 72^\circ = 60 \times 1.0515 = 63.09 \text{ sq. ft.}$$

9. Let the perimeters of pentagon and decagon be denoted by  $10a$  and  $10b$  respectively. Then as in Example 2, page 209,

$$\text{area of pentagon} = 5a^2 \cot \frac{\pi}{5},$$

$$\text{area of decagon} = \frac{10b^2}{4} \cot \frac{\pi}{10};$$

$$\therefore 2a^2 \cot 36^\circ = b^2 \cot 18^\circ.$$

$$\text{Now} \quad \cot^2 18^\circ = \frac{1 + \cos 36^\circ}{1 - \cos 36^\circ} = \frac{1 + \frac{\sqrt{5}+1}{4}}{1 - \frac{\sqrt{5}+1}{4}} = \frac{5 + \sqrt{5}}{3 - \sqrt{5}};$$

$$\text{and} \quad \cot^2 36^\circ = \frac{1 + \frac{\sqrt{5}-1}{4}}{1 - \frac{\sqrt{5}-1}{4}} = \frac{3 + \sqrt{5}}{5 - \sqrt{5}};$$

$$\therefore \frac{\cot^2 18^\circ}{\cot^2 36^\circ} = \frac{20}{4} = 5.$$

$$\therefore \frac{a^2}{b^2} = \frac{\sqrt{5}}{2}, \text{ or } \frac{a}{b} = \frac{\sqrt{5}}{\sqrt{2}}.$$

10. Let  $2na$  be the common perimeter, so that  $2a, a$  are sides respectively of the two polygons.

$$\text{Area of polygon of } n \text{ sides} = \frac{n}{4} \cdot 4a^2 \cot \frac{\pi}{n}.$$

$$\text{Area of polygon of } 2n \text{ sides} = \frac{2n}{4} \cdot a^2 \cot \frac{\pi}{2n};$$

$$\begin{aligned} \therefore \text{ratio of areas} &= \left( 2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n} \right) : \left( \sin \frac{\pi}{n} \cos \frac{\pi}{2n} \right) \\ &= \left( 2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n} \right) : \left( 2 \sin \frac{\pi}{2n} \cos^2 \frac{\pi}{2n} \right) \\ &= \left( 2 \cos \frac{\pi}{n} \right) : \left( 1 + \cos \frac{\pi}{n} \right). \end{aligned}$$

11. In the fig. of Art. 214, if  $AD=a$ ,  $OA=R$ , we have  $R=a \operatorname{cosec} \frac{\pi}{n}$ .

In the fig. of Art. 215, if  $OD=r$ , we have  $r=a \cot \frac{\pi}{n}$ .

$$\begin{aligned} \therefore R+r &= a \left( \frac{1}{\sin \frac{\pi}{n}} + \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right) \\ &= \frac{a \left( 1 + \cos \frac{\pi}{n} \right)}{2 \sin \frac{\pi}{2n} \cdot \cos \frac{\pi}{2n}} = \frac{2a \cos^2 \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n} \cdot \cos \frac{\pi}{2n}} = a \cot \frac{\pi}{2n}. \end{aligned}$$

12. Let  $p, h, d$  represent a side of the pentagon, hexagon, and decagon respectively inscribed in a circle of radius  $r$ ; then

$$p=2r \sin 36^\circ, \quad h=2r \sin 30^\circ, \quad d=2r \sin 18^\circ.$$

$$\begin{aligned} \therefore h^2+d^2 &= 4r^2 \left\{ \frac{1}{4} + \left( \frac{\sqrt{5}-1}{4} \right)^2 \right\} = 4r^2 \left( \frac{4+6-2\sqrt{5}}{16} \right) \\ &= 4r^2 \left( \frac{10-2\sqrt{5}}{16} \right) = 4r^2 \sin^2 36^\circ = p^2. \quad [\text{See Ex. Art. 126.}] \end{aligned}$$

$$13. \quad A_1 = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}, \quad B_1 = nr^2 \tan \frac{\pi}{n}, \quad [\text{Arts. 214, 215}]$$

$$A_2 = \frac{1}{2} 2n \cdot r^2 \sin \frac{\pi}{n}, \quad B_2 = 2nr^2 \tan \frac{\pi}{2n}.$$

$$\therefore A_1 B_1 = \frac{1}{2} n^2 r^4 \cdot 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \tan \frac{\pi}{n} = n^2 r^4 \sin^2 \frac{\pi}{n} = A_2^2.$$

Thus  $A_2$  is the geom. mean between  $A_1$  and  $B_1$ .

$$\begin{aligned} \text{Again } \frac{1}{A_2} + \frac{1}{B_1} &= \frac{1}{nr^2 \sin \frac{\pi}{n}} + \frac{1}{nr^2 \tan \frac{\pi}{n}} = \frac{1 + \cos \frac{\pi}{n}}{nr^2 \sin \frac{\pi}{n}} \\ &= \frac{2 \cos^2 \frac{\pi}{2n}}{2nr^2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}} \cdot \frac{1}{nr^2 \tan \frac{\pi}{2n}} = \frac{2}{B_2}. \end{aligned}$$

Thus  $B_2$  is the harm. mean between  $A_2$  and  $B_1$ .

# EXAMPLES. XVIII. c. PAGE 218.

1. Required distance

$$= r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$2. \quad I_1 A = r_1 \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2},$$

$$I_1 B = r_1 \operatorname{cosec} \left( 90^\circ - \frac{B}{2} \right) = r_1 \sec \frac{B}{2} = 4R \sin \frac{A}{2} \cos \frac{C}{2},$$

$$I_1 C = r_1 \operatorname{cosec} \left( 90^\circ - \frac{C}{2} \right) = r_1 \sec \frac{C}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2}.$$

3. (1) From the fig. of Art. 219, we have

$$\text{area} = \frac{1}{2} I_1 I_2 \cdot I_3 C = \frac{1}{2} 4R \cos \frac{C}{2} \cdot r_3 \operatorname{cosec} \frac{C}{2}$$

$$= 2Rr_3 \cot \frac{C}{2} = 2Rs.$$

$$(2) \quad \text{Area} = 2Rs = 2R \cdot \frac{\Delta}{r} = \frac{1}{2} \Delta \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}. \quad [\text{Art. 212.}]$$

$$4. \quad \text{We have } II_1 = IC \operatorname{cosec} II_1 C = IC \operatorname{cosec} \frac{B}{2}.$$

$$\therefore rII_1 \cdot II_2 \cdot II_3 = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot IA \cdot IB \cdot IC \cdot \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \\ = 4R \cdot IA \cdot IB \cdot IC.$$

$$5. \quad \text{Perimeter of pedal triangle} = R (\sin 2A + \sin 2B + \sin 2C) \\ = 4R \sin A \sin B \sin C.$$

In-radius of pedal triangle

$$= 4 \cdot \frac{R}{2} \sin (90^\circ - A) \sin (90^\circ - B) \sin (90^\circ - C) \\ = 2R \cos A \cos B \cos C.$$

[Arts. 225, 212]

$$6. \quad (1) \quad \frac{g}{a^2} + \frac{h}{b^2} + \frac{k}{c^2} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} \\ = \frac{2bc \cos A + 2ca \cos B + 2ab \cos C}{2abc} \\ = \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}.$$

$$\begin{aligned}
 (2) \quad \frac{b^2 - c^2}{a^2} g + \dots + \dots &= \frac{b^2 - c^2}{a} \cos A + \dots + \dots \\
 &= \frac{(b^2 - c^2)(b^2 + c^2 - a^2)}{2abc} + \dots + \dots \\
 &= 0.
 \end{aligned}$$

7. Let  $\rho_1, \rho_2, \rho_3$  be the radii, then

$$\rho_1 = 4 \frac{R}{2} \cdot \sin \frac{G}{2} \cos \frac{H}{2} \cos \frac{K}{2} = 2R \cos A \sin B \sin C. \quad [\text{Arts. 225, 212.}]$$

8. (1)  $a \cos A, b \cos B, c \cos C, 180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$  are the sides and angles of the pedal triangle of the triangle  $ABC$ . Hence in any formula connecting  $a, b, c, A, B, C$ , we may replace the sides by  $a \cos A, b \cos B, c \cos C$  respectively, and the angles by  $180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$  respectively.

(2)  $a \operatorname{cosec} \frac{A}{2}, b \operatorname{cosec} \frac{B}{2}, c \operatorname{cosec} \frac{C}{2}, 90^\circ - \frac{A}{2}, 90^\circ - \frac{B}{2}, 90^\circ - \frac{C}{2}$  are the sides and angles of the ex-central triangle of the triangle  $ABC$ . Hence the proposition follows.

9. We have  $SI^2 = R^2 - 2Rr$  [Art. 228].

$\therefore R^2 - 2Rr$  must be positive; that is  $R - 2r$  must be positive.

Hence  $R$  can never exceed  $2r$ .

10. If  $R = 2r$ ;

we have  $SI^2 = R^2 - 2Rr = 0$ ;

$\therefore$  the centres of the circumcircle and in-circle coincide, and hence the triangle is equilateral.

$$\begin{aligned}
 11. \quad SI^2 + SI_1^2 + SI_2^2 + SI_3^2 &= R^2 - 2Rr + R^2 + 2Rr_1 + R^2 + 2Rr_2 + R^2 + 2Rr_3 \\
 &= 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\
 &= 12R^2 \quad [\text{XVIII. a. Ex. 25}].
 \end{aligned}$$

$$12. (1) \quad a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = r^2 a \operatorname{cosec}^2 \frac{A}{2} + \dots + \dots$$

$$\begin{aligned}
 &= abc r^2 \left\{ \frac{1}{(s-b)(s-c)} + \dots + \dots \right\} \\
 &= \frac{abc r^2 s}{(s-a)(s-b)(s-c)} = \frac{abc r^2 s^2}{\Delta^2} = abc.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad a \cdot AI_1^2 + b \cdot BI_1^2 + c \cdot CI_1^2 &= r_1^2 \left( a \operatorname{cosec}^2 \frac{A}{2} + b \sec^2 \frac{B}{2} + c \sec^2 \frac{C}{2} \right) \\
 &= abc r_1^2 \left\{ \frac{1}{(s-b)(s-c)} - \frac{1}{s(s-b)} - \frac{1}{s(s-c)} \right\} \\
 &= \frac{abc r_1^2 \{ s - (s-c) - (s-b) \}}{s(s-b)(s-c)} = \frac{abc r_1^2 (s-a)^2}{\Delta^2} = abc.
 \end{aligned}$$

$$13. (1) \text{ We have } \frac{OG}{AG} = \frac{\Delta OBC}{\Delta ABC}.$$

$$\therefore \frac{OG}{AG} + \frac{OH}{BH} + \frac{OK}{CK} = \frac{\text{sum of areas of } \Delta^* OBC, OAC, OAB}{\Delta ABC} = 1.$$

$$(2) \text{ Since } A, H, O, K \text{ are concyclic, } HK = AO \sin A;$$

$$\text{thus } AO = \frac{a \cos A}{\sin A} = a \cot A.$$

$\therefore OG + a \cot A = OG + AO = AG$ , and the result required is reduced to that already proved in (1).

$$14. \text{ Circum-radius of } \Delta AHK = \frac{KH}{2 \sin A} = \frac{R \sin 2A}{2 \sin A} = R \cos A;$$

$$\therefore \text{sum of circum-radii of } \Delta^* AHK, BKG, CGH = R(\cos A + \cos B + \cos C)$$

$$= R \left( 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= R + r.$$

$$15. \text{ We have } A_1 = \frac{\pi}{2} - \frac{A}{2},$$

$$A_2 = \frac{\pi}{2} - \frac{A_1}{2} = \frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \frac{A}{2} \right),$$

$$A_3 = \frac{\pi}{2} - \frac{A_2}{2} = \frac{\pi}{2} - \frac{1}{2} \left\{ \frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \frac{A}{2} \right) \right\};$$

$$\therefore A_n = \frac{\pi}{2} \left( 1 - \frac{1}{2} + \left( \frac{1}{2} \right)^2 - \dots \text{to } n \text{ terms} \right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{2} \frac{1 - \left( -\frac{1}{2} \right)^n}{1 + \frac{1}{2}} + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} \left( 1 - (-1)^n \frac{1}{2^n} \right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left( A - \frac{\pi}{3} \right);$$

similarly  $B_n = \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left( B - \frac{\pi}{3} \right),$

$$C_n = \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left( C - \frac{\pi}{3} \right).$$

When  $n$  is indefinitely increased, then  $A_n = B_n = C_n = \frac{\pi}{3}.$

16. (1)  $OS^2 = R^2 - 8R^2 \cos A \cos B \cos C$  [Art. 230]

$$\begin{aligned} &= R^2 + 2R^2 (1 + \cos 2A + \cos 2B + \cos 2C) \\ &= 9R^2 - 2R^2 (1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C) \\ &= 9R^2 - 4R^2 (\sin^2 A + \sin^2 B + \sin^2 C) \\ &= 9R^2 - a^2 - b^2 - c^2. \end{aligned}$$

(2) We have  $AO = 2R \cos A$ ;  $IA = r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}$ ;

$$\angle IAO = (90^\circ - B) - \frac{A}{2} = \frac{C - B}{2};$$

$$\begin{aligned} \therefore OI^2 &= 4R^2 \cos^2 A + 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 16R^2 \cos A \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{C - B}{2} \\ &= 4R^2 \left( \cos^2 A + 4 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - \cos A \sin B \sin C - 4 \cos A \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \right) \\ &= 4R^2 \left( \cos^2 A + 8 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\ &= 2r^2 - 4R^2 \cos A (\sin B \sin C - \cos A) \\ &= 2r^2 - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

(3) We have

$$AO = 2R \cos A; \quad I_1A = r_1 \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2}; \quad \angle I_1AO = \frac{C - B}{2};$$

$$\begin{aligned} \therefore OI_1^2 &= 4R^2 \cos^2 A + 16R^2 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 16R^2 \cos A \cos \frac{B}{2} \cos \frac{C}{2} \cos \frac{C - B}{2} \\ &= 4R^2 \left( \cos^2 A + 4 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 4 \cos A \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\ &= 4R^2 \left( \cos^2 A + 8 \sin^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\ &= 2r_1^2 - 4R^2 \cos A (\sin B \sin C - \cos A) \\ &= 2r_1^2 - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

17. Let  $R'$ ,  $\rho$ ,  $\rho_1$ ,  $\rho_2$ ,  $\rho_3$  represent the circum-radius, the in-radius, and the ex-radii of the pedal triangle; then, since  $A$ ,  $B$ ,  $C$  are the ex-centres of the pedal triangle, we have, as in Art. 229,

$$f^2 = R'^2 + 2R'\rho_1; \text{ also } R' = \frac{R}{2}.$$

$$\therefore f^2 + g^2 + h^2 = 3R'^2 + 2R'(\rho_1 + \rho_2 + \rho_3) \\ = 3R'^2 + 2R'(4R' + \rho) \quad [\text{XVIII. a. Ex. 25}]$$

$$= 11R'^2 + 2R'\rho$$

$$= 11R'^2 + 8R'^2 \sin \frac{G}{2} \sin \frac{H}{2} \sin \frac{F}{2}$$

$$= 11R'^2 + 8R'^2 \cos A \cos B \cos C; \quad [\text{Art. 224}]$$

$$\therefore 4(f^2 + g^2 + h^2) = 11R^2 + 8R^2 \cos A \cos B \cos C. \quad [\text{Art. 225.}]$$

### EXAMPLES. XVIII. d. PAGE 223.

1. Let  $r$  be the radius, and  $a$ ,  $b$ ,  $c$ ,  $d$  the sides of the quadrilateral. Then we have  $2S = ra + rb + rc + rd$ ;

$$\therefore r = \frac{S}{s}.$$

2. Let  $ABCD$  be the quadrilateral having sides

$$AB = BC = 3; \quad CD = DA = 4.$$

Then  $\angle BAC = \angle BCA; \quad \angle DAC = \angle DCA;$

$$\angle BAD + \angle BCD = 180^\circ; \quad \therefore \angle BAD = \angle BCD = 90^\circ,$$

and the  $\Delta^s BAD, BCD$  are identically equal.

Thus it easily follows that the bisectors of the  $\angle^s BAD, BCD$  meet on  $BD$  which bisects the angles  $ABC, ADC$ .

$\therefore$  a circle can be inscribed in the quadrilateral.

If  $r$  be its radius we have

$$r(3 + 3 + 4 + 4) = 2(\text{area of quadrilateral}) = 2 \cdot 4 \cdot 3.$$

$$\therefore r = \frac{12}{7} = 1\frac{5}{7}.$$

$$\text{Also radius of circumscribed circle} = \frac{1}{2}BD = \frac{1}{2}\sqrt{3^2 + 4^2} = \frac{5}{2} = 2\frac{1}{2}.$$

3. Let  $ABCD$  be the quadrilateral, and let

$$AB=1, BC=2, CD=4, DA=5.$$

Then

$$AC^2 = 5 - 4 \cos ADC = 5 + 4 \cos ADC.$$

also

$$AC^2 = 25 - 24 \cos ADC;$$

$$\therefore 5 + 4 \cos ADC = 25 - 24 \cos ADC.$$

$$\therefore \cos ADC = \frac{20}{28} = \frac{5}{7};$$

$$\text{and area of quadrilateral} = \sqrt{4 \times 3 \times 1 \times 2} = \sqrt{24};$$

$$\therefore \text{radius of inscribed circle} = \frac{\sqrt{24}}{5} = .98 \text{ nearly.}$$

4. Let  $ABCD$  be the quadrilateral, and let

$$AB=60, BC=25, CD=52, DA=39.$$

$$\text{Then } AC^2 = 60^2 + 25^2 - 2 \cdot 60 \cdot 25 \cos ABC = 52^2 + 39^2 - 2 \times 52 \times 39 \cos ADC;$$

$$\text{also } AC^2 = 52^2 + 39^2 - 2 \times 52 \times 39 \cos ADC = 52^2 + 39^2 + 2 \times 52 \times 39 \cos ABC;$$

$$\therefore \cos ABC = 0, \text{ that is } \angle ABC = 90^\circ;$$

and hence the  $\angle ADC = 90^\circ$ ;

$$\therefore AC^2 = 60^2 + 25^2 = 52^2 + 39^2; \therefore AC = 65;$$

$$\therefore BD = \frac{60 \times 52 + 39 \times 25}{65} = 48 + 15 = 63.$$

$$\text{Also the area} = \sqrt{28 \times 63 \times 36 \times 49} = 6^2 \times 7^2 = 1764.$$

5. Let  $ABCD$  be the quadrilateral, and let

$$AB=4, BC=5, CD=8, DA=9.$$

Then since  $AB+BC=9$ ,  $AC$  must be less than 9;

$$\therefore \text{diagonal } BD = 9;$$

and

$$\cos A = \frac{2}{9}; \therefore \sin A = \frac{\sqrt{77}}{9};$$

$$\cos C = \frac{1}{10}; \therefore \sin C = \frac{3\sqrt{11}}{10};$$

$$\therefore \text{area} = \frac{1}{2}(ad \sin A + bc \sin C) = 18 \frac{\sqrt{77}}{9} + 20 \cdot \frac{3\sqrt{11}}{10} = 2\sqrt{77} + 6\sqrt{11}.$$

6. Since the quadrilateral is such that one circle can be inscribed in it and another circle circumscribed about it, therefore

$$\cos A = \frac{ad - bc}{ad + bc}, \text{ and } \sin A = \frac{2\sqrt{abcd}}{ad + bc}; \quad [\text{Art. 234, Ex.}]$$

$$\therefore \text{area} = \frac{1}{2}(ad \sin A + bc \sin C) = \sqrt{abcd}.$$

If  $r$  be radius of inscribed circle,

$$\frac{r}{2}(a + b + c + d) = \text{area} = \sqrt{abcd}.$$

$$\therefore r = \frac{2\sqrt{abcd}}{a + b + c + d}.$$

7. We have  $S^2 = (\sigma - a)(\sigma - b)(\sigma - c)(\sigma - d) - abcd \cos^2 \alpha$ ; [Art. 232]

$\therefore S$  is greatest when  $\cos \alpha = 0$ , since  $\sigma, a, b, c, d$  are constant.

But  $\alpha$  is half the sum of two opposite angles;

$\therefore$  area is a maximum when the sum of two opposite angles is  $180^\circ$ ; that is, when the quadrilateral can be inscribed in a circle.

8.  $23 + 29 + 37 + 41 = 130$ ;

$$\therefore \text{maximum area} = \sqrt{(65 - 23)(65 - 29)(65 - 37)(65 - 41)} \text{ sq. inches} \\ = 6 \times 6 \times 4 \times 7 \text{ sq. inches} = 7 \text{ sq. feet.}$$

9. We have  $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$ ; [Art. 233]

$$\therefore \tan^2 \frac{B}{2} = \frac{1 - \cos B}{1 + \cos B} = \frac{(c + d)^2 - (a - b)^2}{(a + b)^2 - (c - d)^2} \\ = \frac{(a + c + d - b)(c + d + b - a)}{(a + b + c - d)(a + b + d - c)} = \frac{(\sigma - a)(\sigma - b)}{(\sigma - c)(\sigma - d)}.$$

10. See figure on page 220. Let  $\angle DPA = \beta$ ;  
then

$$a^2 = AP^2 + PB^2 + 2AP \cdot PB \cos \beta;$$

$$c^2 = DP^2 + PC^2 + 2DP \cdot PC \cos \beta;$$

$$b^2 = BP^2 + PC^2 - 2BP \cdot PC \cos \beta;$$

$$d^2 = DP^2 + PA^2 - 2DP \cdot PA \cos \beta;$$

$$\therefore (a^2 + c^2) - (b^2 + d^2) = 2 \cos \beta \{AP(DP + PB) + PC(DP + PB)\} \\ = 2AC \cdot BD \cos \beta = 2fg \cos \beta.$$

$$11. \text{ Area} = \frac{1}{2} AC \cdot BD \sin \beta \quad [\text{Art. 231}]$$

$$= \frac{1}{4} \{(a^2 + c^2) - (b^2 + d^2)\} \tan \beta. \quad [\text{Ex. 10.}]$$

$$12. \text{ We have } a + c = b + d; \therefore (a - d)^2 = (b - c)^2.$$

$$\text{Also } a^2 + d^2 - 2ad \cos A = b^2 + c^2 - 2bc \cos C;$$

$$\therefore \text{ by subtraction, } ad(1 - \cos A) = bc(1 - \cos C);$$

$$\text{that is, } ad - bc = ad \cos A - bc \cos C;$$

$$\therefore a^2 d^2 (1 - \cos^2 A) + b^2 c^2 (1 - \cos^2 C) = 2abcd - 2abcd \cos A \cos C;$$

$$\text{or } a^2 d^2 \sin^2 A + b^2 c^2 \sin^2 C = 2abcd - 2abcd \cos A \cos C.$$

$$\therefore (2S)^2 - 2abcd \sin A \sin C = 2abcd - 2abcd \cos A \cos C.$$

$$\therefore 4S^2 = 2abcd (1 - \cos(A + C)) = 4abcd \sin^2 \frac{A + C}{2}.$$

$$\therefore S = \sqrt{abcd} \sin \frac{A + C}{2}.$$

13. If  $\beta$  be the angle between the diagonals, we have

$$S = \frac{1}{2} fg \sin \beta.$$

$$\therefore f^2 g^2 - 4S^2 = f^2 g^2 \cos^2 \beta$$

$$= \frac{1}{4} \{(a^2 + c^2) - (b^2 + d^2)\}^2 \quad [\text{Ex. 10}]$$

$$= \frac{1}{4} (2bd - 2ac)^2, \text{ since } a + c = b + d;$$

$$\text{that is, } 4S^2 = f^2 g^2 - (ac - bd)^2.$$

14. (1) By Euc. vi. 13, we have

$$(ac + bd) \sin \beta = fg \sin \beta = 2S = (ad + bc) \sin A.$$

$$(2) \cos \beta = \frac{(a^2 + c^2) - (b^2 + d^2)}{2fg} \quad [\text{Ex. 10}]$$

$$= \frac{(a^2 + c^2) - (b^2 + d^2)}{2(ac + bd)}.$$

$$\begin{aligned}
 (3) \quad \tan^2 \frac{\beta}{2} &= \frac{1 - \cos \beta}{1 + \cos \beta} = \frac{(b+d)^2 - (c-a)^2}{(c+a)^2 - (b-d)^2}, \text{ or } \frac{(c+a)^2 - (b-d)^2}{(b+d)^2 - (c-a)^2} \\
 &= \frac{(b+d+c-a)(b+d-c+a)}{(c+a-b+d)(c+a+b-d)}, \text{ or } \frac{(c+a-b+d)(c+a+b-d)}{(b+d+c-a)(b+d-c+a)} \\
 &= \frac{(\sigma-a)(\sigma-c)}{(\sigma-b)(\sigma-d)}, \text{ or } \frac{(\sigma-b)(\sigma-d)}{(\sigma-a)(\sigma-c)}.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad S &= \frac{1}{2}fg \sin \beta \\
 &= \frac{1}{4} \sqrt{4f^2g^2 - 4f^2g^2 \cos^2 \beta} \\
 &= \frac{1}{4} \sqrt{4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2}. \quad [\text{Ex. 10.}]
 \end{aligned}$$

16. See figure on page 220.

We have  $\frac{AP}{PC} = \frac{\triangle APB}{\triangle CPB} = \frac{\triangle APD}{\triangle CPD} = \frac{\triangle PAB}{\triangle PCB} = \frac{ad}{bc};$

$$\therefore AP = \frac{ad}{ad+bc} \cdot AC;$$

$$PC = \frac{bc}{ad+bc} \cdot AC;$$

$$\begin{aligned}
 \therefore AP \cdot PC &= \frac{abcd}{(ad+bc)^2} \cdot AC^2 = \frac{abcd}{(ad+bc)^2} \cdot \frac{(ad+bc)(ac+bd)}{ab+cd} \\
 &= \frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}.
 \end{aligned}$$

### EXAMPLES. XVIII. e. PAGE 225.

1.

$$242 + 1212 + 1450 = 2904;$$

$$\begin{aligned}
 \therefore \text{area} &= \sqrt{1452 \times 1210 \times 240 \times 2} \text{ sq. yds.} \\
 &= 8 \times 3 \times 121 \times 10 \text{ sq. yds.} \\
 &= \frac{8 \times 3 \times 121 \times 10}{4840} = 6 \text{ acres.}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \text{Area} &= \frac{200^2 \sin 22\frac{1}{2}^\circ \sin 67\frac{1}{2}^\circ}{2 \sin (22\frac{1}{2}^\circ + 67\frac{1}{2}^\circ)} = \frac{200^2 \cos 45^\circ}{4} \\
 &= \frac{10000 \sqrt{2}}{2} = \frac{14142}{2} = 7071 \text{ sq. yds.}
 \end{aligned}$$

3. We have  $\frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{2\Delta}{s-c}$ ;

$$\therefore b=c, \text{ and } s-b=2(s-a);$$

$$\therefore \frac{c+a-b}{2} = b+c-a;$$

that is,

$$\frac{a}{2} = 2b-a;$$

that is,

$$3a=4b.$$

4. If  $a, b, c$  are in A.P., we have  $a+c=2b$ ;

$$\therefore \frac{1}{r_1} + \frac{1}{r_2} = \frac{s-a+s-c}{\Delta} = \frac{2(s-b)}{\Delta} = \frac{2}{r_3};$$

$$\therefore r_1, r_2, r_3 \text{ are in H.P.}$$

5. We have  $s = \frac{z}{x} + \frac{x}{y} + \frac{y}{z}$ .

$$\therefore \text{area} = \sqrt{\left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right) \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = \sqrt{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}.$$

6. We have  $\frac{\sin(A-B)}{\sin(B-C)} = \frac{\sin A}{\sin C} = \frac{\sin(B+C)}{\sin(A+B)}$ ;

$$\therefore \sin(A-B) \sin(A+B) = \sin(B-C) \sin(B+C);$$

or

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C;$$

$$\therefore a^2 - b^2 = b^2 - c^2;$$

that is,  $a^2, b^2, c^2$  are in A.P.

7. First side  $= \frac{a \sin A + b \sin B + c \sin C}{\sin A + \sin B + \sin C} = \frac{a^2 + b^2 + c^2}{a + b + c} = \frac{a^2 + b^2 + c^2}{2s}.$

8. First side  $= \frac{a}{\sin A} \cdot (a+b+c) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$   
 $= \frac{abc}{2\Delta} \cdot 2s \frac{(s-a)(s-b)(s-c)}{abc} = \frac{\Delta^2}{\Delta} = \Delta.$

9. By Ex. 20, XVIII. a. we have

$$\text{first side} = abc \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

$$= abc \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)} \cdot \frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{abc s^2}{\Delta} = 4Rs^2 = 4R(r_2 r_3 + r_3 r_1 + r_1 r_2).$$

[XVIII. a. Ex. 24.]

$$10. \quad \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{r_1}{s} + \frac{r_2}{s} + \frac{r_3}{s} = \frac{r_1 + r_2 + r_3}{s}$$

$$= \frac{r_1 + r_2 + r_3}{(r_2 r_3 + r_3 r_1 + r_1 r_2)^{\frac{1}{2}}}. \quad [\text{XVIII. a. Ex. 24.}]$$

$$11. \quad \text{First side} = bc \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + ca \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + ab \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{s}{\Delta} \{bc(s-a) + ca(s-b) + ab(s-c)\}$$

$$= \frac{s}{\Delta} \{(bc + ca + ab)s - 3abc\}$$

$$= \frac{abc s^2}{\Delta} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{s} \right)$$

$$= 4Rs^2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{s} \right).$$

$$12. \quad \text{First side} = \left( \frac{s + s - a + s - b + s - c}{\Delta} \right)^2 = \left( \frac{2s}{\Delta} \right)^2$$

$$= \frac{4s}{\Delta} \cdot \left( \frac{s - a + s - b + s - c}{\Delta} \right) = \frac{4}{r} \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right).$$

$$13. \quad \text{We have } \frac{\Delta}{s} = 6, \quad 2s = 70; \quad \therefore \Delta = 6 \times 35.$$

Let  $a, b$  be the sides containing the right angle;

$$\text{then} \quad \frac{1}{2}ab = \Delta = 6 \times 35; \quad \therefore ab = 12 \times 35;$$

$$\text{also} \quad a^2 + b^2 = c^2 = \{70 - (a + b)\}^2;$$

$$\therefore 0 = 70^2 - 140(a + b) + 2ab = 70^2 - 140(a + b) + 12 \times 70;$$

$$\therefore a + b = 41 = 20 + 21,$$

$$ab = 12 \times 35 = 20 \times 21;$$

and

$\therefore$  the two sides containing the right angle are 20, 21, and the hypotenuse  
 $= 70 - 41 = 29.$

$$14. \quad \text{It is easily seen from a figure that } \frac{a}{2f} = \tan A;$$

$$\therefore \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = 2(\tan A + \tan B + \tan C)$$

$$= 2 \tan A \tan B \tan C = \frac{abc}{4fgh}.$$

15. Let  $r, r'$  be the radii of the inscribed circles; then since the perimeters of the triangle and hexagon are equal we have

$$6r \tan \frac{\pi}{3} = 12r' \tan \frac{\pi}{6};$$

whence

$$3r = 2r';$$

$$\therefore \frac{\pi r^2}{\pi r'^2} = \frac{4}{9};$$

$\therefore$  areas of inscribed circles are as 4 to 9.

$$16. \text{ Perimeter} = R (\sin 2A + \sin 2B + \sin 2C) = 4R \sin A \sin B \sin C$$

$$= 4R \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{2R^2}.$$

$$17. \text{ Area} = \frac{1}{2} 4R \cos \frac{B}{2} 4R \cos \frac{C}{2} \sin \left( 90^\circ - \frac{A}{2} \right)$$

$$= 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 2R^2 (\sin A + \sin B + \sin C)$$

$$= R(a + b + c) = \frac{abc(a + b + c)}{4\Delta}.$$

18. Let  $O_1, O_2$  be the two circumcentres; then  $O_1O_2$  is at right angles to  $AC$  at its middle point. Draw  $O_1N_1, O_2N_2$  perpendicular to  $AB_1$ ;

then 
$$O_1O_2 = N_1N_2 \operatorname{cosec} A = \frac{c_1 + c_2}{2 \sin A}.$$

19. Let  $f, g$  be the diagonals; then by Euc. vi. D.,

$$(ac + bd) \sin \beta = fg \sin \beta = 2S.$$

[Art. 231.]

$$\therefore \sin \beta = \frac{2S}{ac + bd}.$$

$$20. \text{ We have } r \cdot II_1 = r \cdot IC \operatorname{cosec} II_1C = ICr \operatorname{cosec} \frac{B}{2} = IC \cdot IB.$$

$$\therefore r^3 II_1 \cdot II_2 \cdot II_3 = IA^2 \cdot IB^2 \cdot IC^2.$$

21. Sum of squares of sides of ex-central  $\Delta$

$$= 16R^2 \left( \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right)$$

$$= 16R^2 \left( 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

[XII. d. Ex. 13]

$$= 32R^2 + 32R^2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 32R^2 + 8Rr = 8R(4R + r).$$

22. Since a circle can be inscribed in the quadrilateral,

$$\therefore a + c = b + d;$$

and since a circle can be circumscribed about the quadrilateral,

$$\begin{aligned} \therefore \cos \beta &= \frac{(a^2 + c^2) \sim (b^2 + d^2)}{2(ac + bd)} & [\text{XVIII. d. Ex. 14}] \\ &= \frac{(a + c)^2 \sim (b + d)^2 + 2(ac \sim bd)}{2(ac + bd)} \\ &= \frac{ac \sim bd}{ac + bd}. \end{aligned}$$

23. (1) We have  $2l^2 + 2 \cdot \frac{a^2}{4} = b^2 + c^2;$

$$2m^2 + 2 \cdot \frac{b^2}{4} = c^2 + a^2;$$

$$2n^2 + 2 \cdot \frac{c^2}{4} = a^2 + b^2;$$

$\therefore$  by addition,  $4(l^2 + m^2 + n^2) = 3(a^2 + b^2 + c^2).$

$$\begin{aligned} (2) \text{ First side} &= \frac{1}{4} \{ (b^2 - c^2)(2b^2 + 2c^2 - a^2) + \text{two similar terms} \} \\ &= \frac{1}{4} \{ \frac{1}{2}(b^4 - c^4) - a^2(b^2 - c^2) + \dots + \dots \} \\ &= 0. \end{aligned}$$

$$\begin{aligned} (3) \quad 4l^2 &= 2b^2 + 2c^2 - a^2; \\ \therefore 16(l^4 + m^4 + n^4) &= (2b^2 + 2c^2 - a^2)^2 + \dots + \dots \\ &= 9(a^4 + b^4 + c^4), \text{ on reduction.} \end{aligned}$$

24. We have  $r_1 + r_2 + r_3 = 4R + r$  [XVIII. a. Ex. 25],  
 $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$  [XVIII. a. Ex. 24],  
 $r_1 r_2 r_3 = r s^2;$  [XVIII. a. Ex. 15]

$\therefore$  the equation whose roots are  $r_1, r_2, r_3$  is  
 $x^3 - (4R + r)x^2 + s^2x - rs^2 = 0.$

25. Let  $PQ$  be the tangent to inscribed circle parallel to  $BC$ , and draw  $AHD$  at right angles to  $PQ$  and  $BC$ ;

then 
$$\begin{aligned} \frac{\Delta_1}{\Delta} &= \frac{AP \cdot AQ}{AB \cdot AC} = \frac{AH}{AB} \cdot \frac{AH}{AC} = \left( \frac{AD - 2r}{AD} \right)^2 \\ &= \left( 1 - \frac{2\Delta}{s} \cdot \frac{a}{2\Delta} \right)^2 = \frac{(s - a)^2}{s^2}. \end{aligned}$$

$$\begin{aligned} \therefore \frac{\Delta_1}{(s - a)^2} &= \frac{\Delta}{s^2} \\ &= \frac{\Delta_2}{(s - b)^2} = \frac{\Delta_3}{(s - c)^2}, \text{ by symmetry.} \end{aligned}$$

Otherwise. Let the sides and perimeter of triangle  $APQ$  be denoted by  $a_1, b_1, c_1, 2s_1$ ; then, by Art. 213,  $s_1 = s - a$ .

But 
$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \frac{s}{s_1}; \therefore \frac{\Delta_1}{s_1^2} = \frac{\Delta}{s^2}. \quad [\text{Euc. vi. 19}].$$

That is, 
$$\frac{\Delta_1}{(s-a)^2} = \frac{\Delta}{s^2}.$$

26.  $MN$  is perpendicular to the bisector of the angle  $A$ ;

$\therefore MN$  is parallel to  $I_2I_3$ ;

thus the sides of the  $\triangle LMN$  are parallel to the sides of the excentral  $\Delta$  and the triangles are similar.

Also 
$$MN = 2(s-a) \sin \frac{A}{2};$$

$$\begin{aligned} \therefore \frac{\triangle LMN}{\triangle I_1I_2I_3} &= \frac{MN \cdot NL}{I_2I_3 \cdot I_3I_1} = \frac{2(s-a) \sin \frac{A}{2} \cdot 2(s-b) \sin \frac{B}{2}}{4R \cos \frac{A}{2} \cdot 4R \cos \frac{B}{2}} \\ &= \frac{(s-a)(s-b)(s-c)}{4R^2s} = \frac{\Delta^2}{4R^2s^2} = \frac{r^2}{4R^2}. \end{aligned}$$

27. We have  $\angle PBC = \angle PCB = A$ ;  $\therefore \angle QPR = 180^\circ - 2A$ .

Similarly  $\angle PQR = 180^\circ - 2B$ ,  $\angle QRP = 180^\circ - 2C$ .

Again 
$$AQ = \frac{AC \sin B}{\sin \angle AQC} = \frac{b \sin B}{\sin 2B} = \frac{b}{2 \cos B};$$

similarly 
$$AR = \frac{c}{2 \cos C}.$$

$$\therefore QR = \frac{b}{2 \cos B} + \frac{c}{2 \cos C} = \frac{b \cos C + c \cos B}{2 \cos B \cos C} = \frac{a}{2 \cos B \cos C}.$$

28. (1) We have  $pc \sin \frac{A}{2} + pb \sin \frac{A}{2} = 2\Delta = bc \sin A$ ;

$$\therefore p(b+c) = 2bc \cos \frac{A}{2};$$

$$\therefore \frac{1}{p} \cos \frac{A}{2} = \frac{b+c}{2bc} = \frac{1}{2} \left( \frac{1}{b} + \frac{1}{c} \right);$$

similarly 
$$\frac{1}{q} \cos \frac{B}{2} = \frac{1}{2} \left( \frac{1}{c} + \frac{1}{a} \right), \quad \frac{1}{r} \cos \frac{C}{2} = \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right).$$

$$\therefore \frac{1}{p} \cos \frac{A}{2} + \frac{1}{q} \cos \frac{B}{2} + \frac{1}{r} \cos \frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$$\begin{aligned}
 (2) \quad \text{We have } pqr &= \frac{8a^2b^2c^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{(b+c)(c+a)(a+b)} \\
 &= \frac{2a^2b^2c^2 (\sin A + \sin B + \sin C)}{(b+c)(c+a)(a+b)} \\
 &= \frac{4\Delta abc (a+b+c)}{(b+c)(c+a)(a+b)}; \\
 \therefore \frac{pqr}{4\Delta} &= \frac{abc (a+b+c)}{(b+c)(c+a)(a+b)}.
 \end{aligned}$$

29. Let  $O$  be the orthocentre; then  $G, H, K$  are the middle points of  $OL, OM, ON$  respectively; therefore the sides of the  $\triangle LMN$  are double the sides of the pedal triangle and parallel to them.

Therefore also the angles are equal to the angles of the pedal triangle.

$$\begin{aligned}
 (1) \quad \text{Area of } \triangle LMN &= \frac{1}{2} \cdot 2a \cos A \cdot 2b \cos B \cdot \sin (180^\circ - 2C) \\
 &= 4ab \sin C \cdot \cos A \cos B \cos C = 8\Delta \cos A \cos B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad AL &= AG + OG = c \sin B + c \cos B \cot C \\
 &= \frac{c \cos (B - C)}{\sin C} = \frac{a \cos (B - C)}{\sin A};
 \end{aligned}$$

$$\begin{aligned}
 \therefore AL \sin A + BM \sin B + CN \sin C &= a \cos (B - C) + b \cos (C - A) + c \cos (A - B) \\
 &= 2R \{ \sin B + C \cos B - C + \dots + \dots \} \\
 &= R [(\sin 2B + \sin 2C) + \dots + \dots] \\
 &= 2R [\sin 2A + \sin 2B + \sin 2C] \\
 &= 8R \sin A \sin B \sin C.
 \end{aligned}$$

30. Let  $P$  be the centre of the circle inscribed between the in-circle and the sides  $AB, AC$ ; then

$$(1) \quad \frac{r - r_a}{r + r_a} = \sin \frac{A}{2};$$

$$\therefore r_a = r \cdot \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}} = r \tan^2 \frac{\pi - A}{4}. \quad [\text{Compare XI. t. Ex. 15.}]$$

$$(1) \quad \text{Also} \quad \frac{\pi - A}{4} + \frac{\pi - B}{4} + \frac{\pi - C}{4} = \frac{\pi}{2};$$

$$\begin{aligned}
 \therefore \sqrt{r_b r_c} + \sqrt{r_c r_a} + \sqrt{r_a r_b} &= r \left( \tan \frac{\pi - B}{4} \tan \frac{\pi - C}{4} + \dots + \dots \right) \\
 &= r.
 \end{aligned}$$

31. It is easily seen that the triangle  $XYZ$  is the same as the triangle  $PQR$  in Ex. 27.

$$\begin{aligned}\therefore \text{Perimeter} &= \frac{a}{2 \cos B \cos C} + \frac{b}{2 \cos C \cos A} + \frac{c}{2 \cos A \cos B} \\ &= \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2 \cos A \cos B \cos C} = 2R \tan A \tan B \tan C.\end{aligned}$$

$$\begin{aligned}\text{Area} &= \frac{1}{2} \cdot \frac{a}{2 \cos B \cos C} \cdot \frac{b}{2 \cos C \cos A} \cdot \sin 2C \\ &= \frac{ab \sin C}{4 \cos A \cos B \cos C} = R^2 \tan A \tan B \tan C.\end{aligned}$$

32. Let  $XYZ$  be the triangle formed by the tangents, and let the perpendiculars from  $X, Y, Z$  to the chord be represented by  $x, y, z$  respectively.

Then the area of

$$\begin{aligned}\Delta XYZ &= \Delta BXC - \Delta ACY + \Delta AZB \\ &= \frac{1}{2} x \cdot BC - \frac{1}{2} y \cdot AC + \frac{1}{2} z \cdot AB.\end{aligned}$$

Now  $\frac{x}{CX} = \frac{DC}{CO}$ , since  $OCX$  is a right angle.

Also  $\frac{CX}{CO} = \frac{DB}{OD} = \frac{DR}{p}$ , since  $O, B, C, X$  are concyclic.

$$\therefore x = \frac{DC \cdot DB}{p}.$$

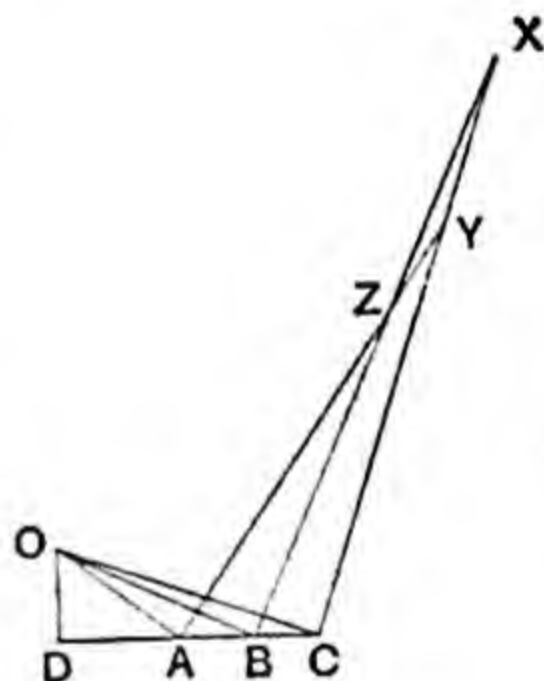
$\therefore \Delta BXC = \frac{1}{2p} \cdot DC \cdot DB (DC - DB) = \frac{cb(c-b)}{2p}$ , if  $a, b, c$  denote the distances of  $A, B, C$  from  $D$ .

$$\therefore \text{area of } \Delta XYZ = \frac{cb(c-b) - ca(c-a) + ba(b-a)}{2p}$$

$$= \frac{cb(c-b) + ba(b-a) + ac(a-c)}{2p}$$

$$= -\frac{(c-b)(b-a)(a-c)}{2p} = \frac{BC \cdot CA \cdot AB}{2p},$$

for  $CA = c - a = -(a - c)$ .



## MISCELLANEOUS EXAMPLES. F. PAGE 228.

1. We have  $\cos(\alpha + \beta) = \cos(180^\circ - \gamma + \delta) = -\cos(\gamma + \delta)$ ;  
 $\therefore \cos \alpha \cos \beta - \sin \alpha \sin \beta = \sin \gamma \sin \delta - \cos \gamma \cos \delta$ ;  
 $\therefore \cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta$ .

$$2. \text{ First side} = \frac{\cos(15^\circ - A) \sin 15^\circ - \sin(15^\circ - A) \cos 15^\circ}{\sin 15^\circ \cos 15^\circ}$$

$$= \frac{2 \sin A}{\sin 30^\circ} = 4 \sin A.$$

$$3. \cot A + \sin A \operatorname{cosec} B \operatorname{cosec} C = \frac{\cos A \sin B \sin C + 1 - \cos^2 A}{\sin A \sin B \sin C}$$

$$= \frac{\cos A \{ \sin B \sin C + \cos(B + C) \} + 1}{\sin A \sin B \sin C}$$

$$= \frac{\cos A \cos B \cos C + 1}{\sin A \sin B \sin C},$$

which is symmetrical with respect to  $A, B, C$ .

$$4. \text{ We have } \sin B = \frac{b \sin A}{a} = \frac{\sqrt{8} \cdot \sin 30^\circ}{2} = \frac{1}{\sqrt{2}};$$

$$\therefore B = 45^\circ, \text{ or } 135^\circ;$$

$$\therefore C = 105^\circ, \text{ or } 15^\circ;$$

[Art. 148. (iii)];

$$\therefore c = \frac{a \sin C}{\sin A} = 4 \cos 15^\circ, \text{ or } 4 \sin 15^\circ;$$

$$\therefore c = \sqrt{6} + \sqrt{2}, \text{ or } \sqrt{6} - \sqrt{2}.$$

$$5. (1) \text{ We have } \cot 18^\circ \tan 36^\circ = \frac{2 \cos 18^\circ \sin 36^\circ}{2 \sin 18^\circ \cos 36^\circ}$$

$$= \frac{\sin 54^\circ + \sin 18^\circ}{\sin 54^\circ - \sin 18^\circ}$$

$$= \frac{\sqrt{5} + 1 + (\sqrt{5} - 1)}{\sqrt{5} + 1 - (\sqrt{5} - 1)} = \sqrt{5}.$$

$$(2) \sin 36^\circ = \sin 144^\circ, \text{ and } \sin 72^\circ = \sin 108^\circ;$$

$$\therefore \text{ first side} = 4 (2 \sin 72^\circ \sin 36^\circ)^2 = 4 (\cos 36^\circ - \cos 108^\circ)^2$$

$$= 4 (\cos 36^\circ + \sin 18^\circ)^2 = 4 \left( \frac{\sqrt{5} + 1}{4} + \frac{\sqrt{5} - 1}{4} \right)^2$$

$$= 5.$$

$$\begin{aligned}
 6. \quad & \log 2 = .30103; \therefore \log 4 = .60206 \\
 & \log 3 = .47712; \therefore \log 9 = .95424 \\
 & \log 11 = 1.04139 \\
 & \qquad \qquad \qquad \underline{2.59769}
 \end{aligned}$$

$$\therefore \log .0396 = \bar{2}.59769;$$

$$\therefore \log (.0396)^{90} = -180 + 53.7921 = \bar{127}.7921;$$

$\therefore$  number of ciphers before the first significant digit in  $(.0396)^{90}$  is 126.

7. Let  $P, Q$  be the two positions of the observer;

then  $\angle QPB = 30^\circ, \angle QBP = 45^\circ, PQ = 50$  yards;

$$\therefore PB = \frac{50 \sin 75^\circ}{\sin 45^\circ} = \frac{50 \sqrt{2} (\sqrt{3} + 1)}{2 \sqrt{2}} = 25 (\sqrt{3} + 1) = 68.3 \text{ yds.},$$

$$QB = \frac{50 \sin 30^\circ}{\sin 45^\circ} = 50 \sqrt{2} \cdot \frac{1}{2} = 35.35 \text{ yds.}$$

$$\begin{aligned}
 8. \quad \text{First side} &= 2 + \frac{1}{2} \{ \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2(\alpha + \beta + \gamma) \} \\
 &= 2 + \cos(\alpha + \beta) \cos(\alpha - \beta) + \cos(\alpha + \beta + 2\gamma) \cos(\alpha + \beta) \\
 &= 2 + 2 \cos(\alpha + \beta) \cos(\beta + \gamma) \cos(\gamma + \alpha).
 \end{aligned}$$

$$\begin{aligned}
 9. \quad (1) \quad \tan 40^\circ + \cot 40^\circ &= \frac{\sin^2 40^\circ + \cos^2 40^\circ}{\cos 40^\circ \sin 40^\circ} = \frac{2}{2 \sin 40^\circ \cos 40^\circ} \\
 &= \frac{2}{\sin 80^\circ} = 2 \sec 10^\circ.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \tan 70^\circ + \tan 20^\circ &= \tan 20^\circ + \cot 20^\circ = \frac{2}{\sin 40^\circ}, \text{ as in (1),} \\
 &= 2 \operatorname{cosec} 40^\circ.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad (1) \quad \text{First side} &= 2 \sin 4\alpha - 2 \cos 6\alpha \sin 4\alpha \\
 &= 2 \sin 4\alpha (1 - \cos 6\alpha) = 4 \sin 4\alpha \sin^2 3\alpha \\
 &= 16 \sin \alpha \cos \alpha \cos 2\alpha \sin^2 3\alpha.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{First side} &= \sin \frac{2\pi}{7} - 2 \cos \frac{5\pi}{7} \sin \frac{\pi}{7} \\
 &= 2 \sin \frac{\pi}{7} \left( \cos \frac{\pi}{7} - \cos \frac{5\pi}{7} \right) \\
 &= 4 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7} \\
 &= 4 \sin \frac{\pi}{7} \sin \frac{5\pi}{7} \sin \frac{3\pi}{7}.
 \end{aligned}$$

$$11. \text{ We have } \sin C = \frac{c \sin B}{b} = \frac{6-2\sqrt{3}}{2(3\sqrt{2}-\sqrt{6})} = \frac{1}{\sqrt{2}}; \quad [\text{Art. 148, (iii)}];$$

$$\therefore C = 45^\circ \text{ or } 135^\circ;$$

$$\therefore A = 105^\circ \text{ or } 15^\circ;$$

$$\therefore a = \frac{c \sin A}{\sin C} = \frac{(6-2\sqrt{3})\sqrt{2}(\sqrt{3}+1)}{2\sqrt{2}}, \text{ or } \frac{(6-2\sqrt{3})\sqrt{2}(\sqrt{3}-1)}{2\sqrt{2}};$$

$$\text{that is, } a = 2\sqrt{3}, \text{ or } 4\sqrt{3}-6.$$

12. Let  $C$  be the rock and  $A, B$  the two positions of the ship.

Then we have

$$\angle BAC = \angle BCA = 67\frac{1}{2}^\circ;$$

$$\therefore BC = BA = 10 \text{ miles,}$$

$$\text{and } AC = 2AB \sin 22\frac{1}{2}^\circ = 10\sqrt{2}-\sqrt{2} \text{ miles.} \quad [\text{Art. 251.}]$$

$$13. \text{ First side } = \frac{\sin^2 B - \sin^2 C}{\cos B + \cos C} + \dots + \dots$$

$$= \frac{\cos^2 C - \cos^2 B}{\cos B + \cos C} + \dots + \dots$$

$$= (\cos C - \cos B) + \dots + \dots$$

$$= 0.$$

$$14. \text{ We have } \frac{1}{\cos(\theta-a)} + \frac{1}{\cos(\theta+a)} = \frac{2}{\cos \theta};$$

$$\therefore \frac{4 \cos \theta \cos a}{\cos 2\theta + \cos 2a} = \frac{2}{\cos \theta};$$

$$\therefore 2 \cos^2 \theta \cos a = 2 \cos^2 \theta - 1 + 2 \cos^2 a - 1;$$

$$\therefore \cos^2 \theta (\cos a - 1) = \cos^2 a - 1;$$

$$\therefore \cos^2 \theta = \cos a + 1 = 2 \cos^2 \frac{a}{2};$$

$$\text{whence } \cos \theta = \sqrt{2} \cos \frac{a}{2}.$$

$$15. \text{ We have } \sin a \cos a = \sin^2 \beta$$

$$= \frac{1}{2}(1 - \cos 2\beta);$$

$$\therefore \cos 2\beta = 1 - 2 \sin a \cos a = (\cos a - \sin a)^2$$

$$= 2 \cos^2 \left( \frac{\pi}{4} + a \right).$$

16. See figure of Art. 223.

$$OG = BG \cot BOG = BG \cot C = c \cos B \cot C$$

$$= 2R \cos B \cos C;$$

$$\text{similarly } OH = 2R \cos C \cos A, \quad OK = 2R \cos A \cos B.$$

17. We have  $\cos \theta = \frac{\cos u - e}{1 - e \cos u}$ .

$$\therefore \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - e \cos u - \cos u + e}{1 - e \cos u + \cos u - e} = \frac{(1 + e)(1 - \cos u)}{(1 - e)(1 + \cos u)};$$

$$\therefore \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \cdot \tan \frac{u}{2}.$$

18. The sum of the squares on the sides containing the right angle  
 $= 4(1 + \sin \theta)^2 + 4(1 + \cos \theta)^2 + \cos^2 \theta + \sin^2 \theta + 4\cos \theta(1 + \sin \theta) + 4\sin \theta(1 + \cos \theta)$   
 $= 8 + 8(\sin \theta + \cos \theta) + 4 + 1 + 4(\sin \theta + \cos \theta) + 8\sin \theta \cos \theta$   
 $= 9 + 12(\sin \theta + \cos \theta) + 4(1 + 2\sin \theta \cos \theta)$   
 $= 9 + 12(\sin \theta + \cos \theta) + 4(\sin \theta + \cos \theta)^2$   
 $= \{3 + 2(\sin \theta + \cos \theta)\}^2.$

$$\therefore \text{hypotenuse} = 3 + 2(\sin \theta + \cos \theta).$$

19. In the figure of Art. 227 let  $O$  be the centre of in-circle of  $I_1 I_2 I_3$ , and  $O_1$  the centre of the ex-circle opposite to  $I_1$ . Let  $R'$  be the circum-radius of  $I_1 I_2 I_3$ .

Then, as in Art. 220,  $OO_1 = 4R' \sin \frac{I_2 I_1 I_3}{2}$ ;

but  $\angle I_2 I_1 I_3 = \frac{\pi}{2} - \frac{A}{2}$ ; and  $R' = 2R$ ; [Arts. 221, 222];

$$\therefore OO_1 = 8R \sin \left( \frac{\pi}{4} - \frac{A}{4} \right) = 8R \sin \frac{B + C}{4}.$$

20. The sides of the ex-central triangle of the triangle  $I_1 I_2 I_3$  are

$$4R' \cos \frac{\pi - A}{4}, \quad 4R' \cos \frac{\pi - B}{4}, \quad 4R' \cos \frac{\pi - C}{4}, \quad [\text{Art. 221}],$$

that is,  $8R \cos \frac{B + C}{4}, \quad 8R \cos \frac{C + A}{4}, \quad 8R \cos \frac{A + B}{4}.$

21. We have

$$(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \gamma) = (1 - \cos \alpha)(1 - \cos \beta)(1 - \cos \gamma);$$

$$\therefore \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} = \pm \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2};$$

$$\therefore \text{each expression} = 8 \cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}$$

$$= \pm 8 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \cdot \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$= \pm \sin \alpha \sin \beta \sin \gamma.$$

22. Let  $\alpha, \beta, \gamma, \delta$  be four angles such that  $\alpha + \beta + \gamma + \delta = 180^\circ$ ;

then  $\cos(\alpha + \beta) = \cos(180^\circ - \gamma + \delta) = -\cos(\gamma + \delta)$ ;  
 $\therefore \cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta$ ;

similarly  $\cos \beta \cos \gamma + \cos \delta \cos \alpha = \sin \beta \sin \gamma + \sin \delta \sin \alpha$ ;

$$\cos \gamma \cos \alpha + \cos \beta \cos \delta = \sin \gamma \sin \alpha + \sin \beta \sin \delta$$

$\therefore$  by addition we have the sum of the products of the cosines taken two together equal to the sum of the products of the sines taken two together.

23. (1)  $II_1 \cdot II_2 \cdot II_3 = 4R \sin \frac{A}{2} \cdot 4R \sin \frac{B}{2} \cdot 4R \sin \frac{C}{2}$  [Art. 220]

$$= 16R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 16R^3 r.$$

(2)  $II_1^2 + I_2 I_3^2 = 16R^2 \sin^2 \frac{A}{2} + 16R^2 \cos^2 \frac{A}{2}$  [Arts. 220, 221]

$$= 16R^2.$$

24. (1) Let  $\alpha, \beta, \gamma$  be the angles;

then  $\cos \alpha = \frac{\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - \cos^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{1 + \cos \frac{B+C}{2} \cos \frac{B-C}{2} - \cos^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}}$

$$= \frac{\sin \frac{A}{2} \cos \frac{B-C}{2} + \sin^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \sin \frac{A}{2} = \cos \left( 90^\circ - \frac{A}{2} \right).$$

Thus the angles are  $90^\circ - \frac{A}{2}$ ,  $90^\circ - \frac{B}{2}$ ,  $90^\circ - \frac{C}{2}$ .

(2) Here

$$\cos \alpha = \frac{\sin^2 2B + \sin^2 2C - \sin^2 2A}{2 \sin 2B \sin 2C} = \frac{1 - \cos 2(B+C) \cos 2(B-C) - \sin^2 2A}{2 \sin 2B \sin 2C}$$

$$= \frac{-\cos 2A \cos 2(B-C) + \cos 2A}{2 \sin 2B \sin 2C}$$

$$= \frac{-\cos 2A \{ \cos 2(B-C) - \cos 2(B+C) \}}{2 \sin 2B \sin 2C} = -\cos 2A.$$

Thus the angles are  $180^\circ - 2A$ ,  $180^\circ - 2B$ ,  $180^\circ - 2C$ .

25. The expression

$$\begin{aligned}
 &= \{\sin(\theta + \alpha) + \sin(\theta + \beta)\}^2 - 2 \sin(\theta + \alpha) \sin(\theta + \beta) \\
 &\quad - 2 \cos(\alpha - \beta) \sin(\theta + \alpha) \sin(\theta + \beta) \\
 &= \left\{ 2 \sin \frac{2\theta + \alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right\}^2 - 2 \sin(\theta + \alpha) \sin(\theta + \beta) (1 + \cos \alpha - \beta) \\
 &= 4 \cos^2 \frac{\alpha - \beta}{2} \left\{ \sin^2 \frac{2\theta + \alpha + \beta}{2} - \sin(\theta + \alpha) \sin(\theta + \beta) \right\} \\
 &= 2 \cos^2 \frac{\alpha - \beta}{2} \{ 1 - \cos(2\theta + \alpha + \beta) - \cos(\alpha - \beta) + \cos(2\theta + \alpha + \beta) \} \\
 &= 2 \cos^2 \frac{\alpha - \beta}{2} (1 - \cos \alpha - \beta),
 \end{aligned}$$

which is independent of  $\theta$ .

26. See figure on page 220.

Since the quadrilateral is described about a circle,

$$\therefore a + c = b + d; \text{ that is, } a - d = b - c.$$

$$\text{Now} \quad a^2 + d^2 - 2ad \cos A = BD^2 = b^2 + c^2 - 2bc \cos C;$$

$$\therefore (a - d)^2 + 2ad(1 - \cos A) = (b - c)^2 + 2bc(1 - \cos C);$$

$$\therefore ad \sin^2 \frac{A}{2} = bc \sin^2 \frac{C}{2}.$$

27. Let the tangent parallel to  $BC$  meet  $AC$  in  $M$ ; and let  $AG$ , the perpendicular from  $A$  to  $BC$ , meet the tangent in  $X$ ; then

$$\frac{p}{a} = \frac{AM}{AC} = \frac{AX}{AG} = \frac{AG - 2r}{AG}$$

$$= 1 - \frac{2r}{AG} = 1 - \frac{2r}{s};$$

$$\therefore \frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 - \frac{2r}{s} + 1 - \frac{2r}{s} + 1 - \frac{2r}{s}$$

$$= 3 - \frac{6r}{s} = 1.$$

28. Take the figure of Art. 199. Then we have

$$\angle PAB = 13^\circ 14' 12'', \quad \angle PBC = 56^\circ 24' 36'', \quad \angle BPA = 43^\circ 10' 24''.$$

Also  $PC = 1566$ . Let  $AB = x$  ft.

$$x = \frac{PB \sin 43^\circ 10' 24''}{\sin 13^\circ 14' 12''}$$

$$\log \sin 43^\circ 10' = \bar{1}.8351341$$

$$\frac{24}{60} \times 1347 = 539$$

$$\log PB = 3.2741376$$

$$3.1093256$$

$$\log \sin 13^\circ 14' 12'' = \bar{1}.3597858$$

$$\log x = 3.7495398$$

$$\log 5617.4 = 3.7495353$$

$$45$$

$$6 \quad 46$$

$$PB = 1566 \operatorname{cosec} 56^\circ 24' 36''.$$

$$\log \operatorname{cosec} 56^\circ 24' = .0793961$$

$$\text{subtract } \frac{36}{60} \times 839 = 503$$

$$\bar{.0793458}$$

$$\log 1566 = 3.1947918$$

$$\log PB = 3.2741376$$

$$\log \sin 13^\circ 14' = \bar{1}.3596785$$

$$\frac{12}{60} \times 5369 = 1073$$

$$1.3597858$$

Thus  $x = 5617.46$  ft., whence it easily follows that the speed of the train is 21.3 miles per hour.

29. Let  $A$  represent the harbour,  $C$  the fort,  $B$  the position of the ship when 20 miles from  $C$ .

Then  $AC = 27.23$  miles,  $CB = 20$  miles,  $\angle CAB = 46^\circ 8' 8.6''$ .

$$\sin B = \frac{27.23 \sin 46^\circ 8' 8.6''}{20}$$

$$\log 27.23 = 1.4350476$$

$$\log \sin 46^\circ 8' = \bar{1}.8579078$$

$$\frac{86}{600} \times 1215 = 174$$

$$1.2929728$$

$$\log 20 = 1.3010300$$

$$\log \sin B = \bar{1}.9919428$$

$$\log \sin 78^\circ 59' = \bar{1}.9919220$$

$$208$$

$$\text{Diff. for } 60'' = 246;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{208}{246} \times 60'' = 50.7''.$$

$\therefore B = 78^\circ 59' 50.7''$ , or  $101^\circ 0' 9.3''$ , both values being admissible since  $a < b$ .

Hence with the third figure of page 131 we have

$$\angle ACB_1 = 54^\circ 52' 0.7'', \quad \angle ACB_2 = 32^\circ 51' 42.1''.$$

In  $\triangle ACB_1$ ,

$$AB_1 = \frac{20 \sin 54^\circ 52' 0.7''}{\sin 46^\circ 8' 8.6''},$$

$$\log \sin 54^\circ 52' = \bar{1}.9126551$$

$$\frac{7}{600} \times 889 = 10$$

$$\log 20 = 1.3010300$$

$$1.2136861$$

$$\log \sin 46^\circ 8' 8.6'' = \bar{1}.8579252$$

$$\log AB_1 = 1.3557609$$

$$\log 22.686 = 1.3557579$$

$$30$$

$$2 \quad 38$$

$$\therefore AB_1 = 22.6862 \text{ miles.}$$

In  $\triangle ACB_2$ ,

$$AB_2 = \frac{20 \sin 32^\circ 51' 42.1''}{\sin 46^\circ 8' 8.6''};$$

$$\log \sin 32^\circ 51' = \bar{1}.7343529$$

$$\frac{421}{600} \times 1956 = 1372$$

$$\log 20 = 1.3010300$$

$$1.0355201$$

$$\log \sin 46^\circ 8' 8.6'' = \bar{1}.8579252$$

$$\log AB_2 = 1.1775949$$

$$\log 15.052 = 1.1775942$$

$$\therefore AB_2 = 15.052 \text{ miles.}$$

Thus the time taken is approximately 2.27 hours or 1.5 hours; that is the ship will be 20 miles from the fort in 2 hrs. 16 min. or in 1 hr. 30 min.

## EXAMPLES. XIX. a. PAGE 235.

$$1. \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6};$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}.$$

$$2. \sin \theta = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4};$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4}.$$

$$3. \cos \theta = \frac{1}{2} = \cos \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

$$4. \tan \theta = \sqrt{3} = \tan \frac{\pi}{3};$$

$$\therefore \theta = n\pi + \frac{\pi}{3}.$$

$$5. \cot \theta = -\sqrt{3} = \cot \left( -\frac{\pi}{6} \right);$$

$$\therefore \theta = n\pi - \frac{\pi}{6}.$$

$$6. \sec \theta = -\sqrt{2} = \sec \frac{3\pi}{4};$$

$$\therefore \theta = 2n\pi \pm \frac{3\pi}{4}.$$

$$7. \cos^2 \theta = \frac{1}{2};$$

$$\therefore \cos \theta = \pm \frac{1}{\sqrt{2}};$$

$$\theta = 2n\pi \pm \frac{\pi}{4}, \text{ or } 2n\pi \pm \left( \pi - \frac{\pi}{4} \right).$$

$$8. \tan^2 \theta = \frac{1}{3};$$

$$\therefore \tan \theta = \pm \frac{1}{\sqrt{3}} = \tan \left( \pm \frac{\pi}{6} \right);$$

$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$

Both of these are included in  $n\pi \pm \frac{\pi}{4}$ .

$$9. \operatorname{cosec}^2 \theta = \frac{4}{3};$$

$$\therefore \cot^2 \theta = \frac{1}{3};$$

$$\therefore \cot \theta = \pm \frac{1}{\sqrt{3}} = \cot \left( \pm \frac{\pi}{3} \right);$$

$$\therefore \theta = n\pi \pm \frac{\pi}{3}.$$

$$12. \sec^2 \theta = \sec^2 \alpha;$$

$$\therefore \tan^2 \theta = \tan^2 \alpha;$$

$$\therefore \theta = n\pi \pm \alpha.$$

$$14. \operatorname{cosec} 3\theta = \operatorname{cosec} 3\alpha;$$

$$\therefore 3\theta = n\pi + (-1)^n 3\alpha;$$

$$\therefore \theta = \frac{n\pi}{3} + (-1)^n \alpha.$$

$$16. \sin 5\theta + \sin \theta = \sin 3\theta;$$

$$\therefore 2 \sin 3\theta \cos 2\theta = \sin 3\theta;$$

$$\therefore \sin 3\theta = 0,$$

$$\text{or} \quad \cos 2\theta = \frac{1}{2},$$

$$\text{whence} \quad \theta = \frac{n\pi}{3}, \text{ or } n\pi \pm \frac{\pi}{6}.$$

18.

$$\sin 4\theta + \sin 2\theta - (\sin 3\theta + \sin \theta) = 0;$$

$$\therefore 2 \sin 3\theta \cos \theta - 2 \sin 2\theta \cos \theta = 0;$$

$$\therefore 2 \cos \theta \cdot 2 \cos \frac{5\theta}{2} \sin \frac{\theta}{2} = 0;$$

$$\therefore \cos \theta = 0, \text{ or } \cos \frac{5\theta}{2} = 0, \text{ or } \sin \frac{\theta}{2} = 0.$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, \text{ or } \frac{(2n+1)\pi}{5}, \text{ or } 2n\pi.$$

19. As in Example 18, we obtain

$$4 \cos \theta \cos 4\theta \cos 2\theta = 0;$$

$$\text{whence} \quad \theta = \frac{(2n+1)\pi}{2}, \text{ or } \frac{(2n+1)\pi}{4}, \text{ or } \frac{(2n+1)\pi}{8}.$$

$$10. \cos \theta = \cos \alpha;$$

$$\therefore \theta = 2n\pi \pm \alpha.$$

$$11. \tan^2 \theta = \tan^2 \alpha;$$

$$\therefore \tan \theta = \pm \tan \alpha = \tan (\pm \alpha);$$

$$\therefore \theta = n\pi \pm \alpha.$$

$$13. \tan 2\theta = \tan \theta;$$

$$\therefore 2\theta = n\pi + \theta;$$

$$\therefore \theta = n\pi.$$

$$15. \cos 3\theta = \cos 2\theta;$$

$$\therefore 3\theta = 2n\pi \pm 2\theta;$$

$$\therefore \theta = 2n\pi, \text{ or } \frac{2n\pi}{5}.$$

$$17. \cos \theta - \cos 7\theta = \sin 4\theta;$$

$$\therefore 2 \sin 4\theta \sin 3\theta = \sin 4\theta;$$

$$\therefore \sin 4\theta = 0, \text{ or } \sin 3\theta = \frac{1}{2};$$

$$\text{whence} \quad \theta = \frac{n\pi}{4}, \text{ or } \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}.$$

20.

$$\begin{aligned}\sin 5\theta \cos \theta &= \sin 6\theta \cos 2\theta; \\ \therefore \sin 6\theta + \sin 4\theta &= \sin 8\theta + \sin 4\theta; \\ \therefore \sin 8\theta - \sin 6\theta &= 0; \\ \therefore 2 \sin \theta \cos 7\theta &= 0; \\ \therefore \sin \theta &= 0, \text{ or } \cos 7\theta = 0;\end{aligned}$$

whence

$$\theta = n\pi, \text{ or } \frac{(2n+1)\pi}{14}.$$

21.

$$\begin{aligned}\sin 11\theta \sin 4\theta + \sin 5\theta \sin 2\theta &= 0; \\ \therefore \cos 7\theta - \cos 15\theta + \cos 3\theta - \cos 7\theta &= 0; \\ \therefore 2 \sin 9\theta \sin 6\theta &= 0;\end{aligned}$$

whence

$$\theta = \frac{n\pi}{9}, \text{ or } \frac{n\pi}{6}.$$

22.

$$\begin{aligned}\sqrt{2} \cos 3\theta - \cos \theta &= \cos 5\theta; \\ \therefore \sqrt{2} \cos 3\theta &= 2 \cos 3\theta \cos 2\theta; \\ \therefore \cos 3\theta &= 0, \text{ or } \cos 2\theta = \frac{1}{\sqrt{2}}; \\ \therefore \theta &= \frac{(2n+1)\pi}{6}, \text{ or } n\pi \pm \frac{\pi}{8}.\end{aligned}$$

23.

$$\begin{aligned}\sin 7\theta - \sqrt{3} \cos 4\theta &= \sin \theta; \\ \therefore 2 \cos 4\theta \sin 3\theta &= \sqrt{3} \cos 4\theta; \\ \therefore \cos 4\theta &= 0, \text{ or } \sin 3\theta = \frac{\sqrt{3}}{2};\end{aligned}$$

whence

$$\theta = \frac{(2n+1)\pi}{8}, \text{ or } \frac{n\pi}{3} + (-1)^n \frac{\pi}{9}.$$

24.

$$\begin{aligned}1 + \cos \theta &= 2 \sin^2 \theta; \\ \therefore 1 + \cos \theta &= 2 - 2 \cos^2 \theta; \\ \therefore 2 \cos^2 \theta + \cos \theta - 1 &= 0; \\ \therefore \cos \theta &= -1, \text{ or } \frac{1}{2}; \\ \therefore \theta &= (2n+1)\pi, \text{ or } 2n\pi \pm \frac{\pi}{3}.\end{aligned}$$

25.

$$\begin{aligned}\tan^2 \theta + \sec \theta &= 1; \\ \therefore \sec^2 \theta + \sec \theta - 2 &= 0; \\ \therefore \sec \theta &= -2, \text{ or } 1; \\ \therefore \theta &= 2n\pi \pm \frac{2\pi}{3}, \text{ or } 2n\pi.\end{aligned}$$

26.

$$\cot^2 \theta - 1 = \operatorname{cosec} \theta;$$

$$\therefore \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta - 2 = 0;$$

$$\therefore \operatorname{cosec} \theta = 2, \text{ or } -1;$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}, \text{ or } n\pi + (-1)^n \frac{3\pi}{2}.$$

27.

$$\cot \theta - \tan \theta = 2;$$

$$\therefore \cot^2 \theta - 1 = 2 \cot \theta;$$

$$\therefore \cot 2\theta = 1;$$

$$\therefore \theta = \frac{n\pi}{2} + \frac{\pi}{8}.$$

28.

$$2 \cos \theta = -1; \therefore \theta = 2n\pi \pm \frac{2\pi}{3} \dots\dots\dots (1),$$

$$2 \sin \theta = \sqrt{3}; \therefore \theta = n\pi + (-1)^n \frac{2\pi}{3} \dots\dots\dots (2).$$

From (1) we see that the multiple of  $\pi$  must be even, and from (2) that the sign before the second term must be positive when the multiple of  $\pi$  is even;

$$\therefore \theta = 2n\pi + \frac{2\pi}{3}.$$

29.

$$\sec \theta = \sqrt{2}; \therefore \theta = 2n\pi \pm \frac{\pi}{4} \dots\dots\dots (1),$$

$$\tan \theta = -1; \therefore \theta = n\pi - \frac{\pi}{4} \dots\dots\dots (2).$$

From (1) we see that the multiple of  $\pi$  must be even, and from (2), that the sign before the second term must be negative;

$$\therefore \theta = 2n\pi - \frac{\pi}{4}.$$

### EXAMPLES. XIX. b. PAGE 237.

1.

$$\tan p\theta = \cot q\theta;$$

$$\therefore \tan p\theta = \tan \left( \frac{\pi}{2} - q\theta \right).$$

$$\therefore p\theta = n\pi + \frac{\pi}{2} - q\theta;$$

$$\therefore \theta = \frac{(2n+1)\pi}{2(p+q)}.$$

2.

$$\sin m\theta + \cos n\theta = 0;$$

$$\therefore \cos n\theta = \cos \left( \frac{\pi}{2} + m\theta \right);$$

$$\therefore n\theta = 2k\pi \pm \left( \frac{\pi}{2} + m\theta \right);$$

$$\therefore \theta = \frac{(4k+1)\pi}{2(n-m)}, \text{ or } \frac{(4k-1)\pi}{2(n+m)}.$$

3.

$$\cos \theta - \sqrt{3} \sin \theta = 1;$$

$$\therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2};$$

$$\therefore \cos \left( \theta + \frac{\pi}{3} \right) = \cos \frac{\pi}{3};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi, \text{ or } 2n\pi - \frac{2\pi}{3}.$$

4.

$$\sin \theta - \sqrt{3} \cos \theta = 1;$$

$$\therefore \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta = \frac{1}{2};$$

$$\therefore \cos \left( \theta + \frac{\pi}{6} \right) = \cos \frac{2\pi}{3};$$

$$\therefore \theta + \frac{\pi}{6} = 2n\pi \pm \frac{2\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi - \frac{5\pi}{6};$$

that is,

$$\theta = 2n\pi + \frac{\pi}{2}, \text{ or } (2n+1)\pi + \frac{\pi}{6}.$$

5.

$$\cos \theta = \sqrt{3} (1 - \sin \theta);$$

$$\therefore \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta = \frac{\sqrt{3}}{2};$$

$$\therefore \cos \left( \theta - \frac{\pi}{3} \right) = \cos \frac{\pi}{6};$$

$$\therefore \theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi + \frac{\pi}{6}.$$

6.

$$\sin \theta + \sqrt{3} \cos \theta = \sqrt{2};$$

$$\therefore \cos \left( \theta - \frac{\pi}{6} \right) = \cos \frac{\pi}{4};$$

$$\therefore \theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4};$$

$$\therefore \theta = 2n\pi + \frac{5\pi}{12}, \text{ or } 2n\pi - \frac{\pi}{12}.$$

7.

$$\cos \theta - \sin \theta = \frac{1}{\sqrt{2}};$$

$$\therefore \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{2},$$

$$\therefore \cos \left( \theta + \frac{\pi}{4} \right) = \cos \frac{\pi}{3};$$

$$\therefore \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{12}, \text{ or } 2n\pi - \frac{7\pi}{12}.$$

8

$$\cos \theta + \sin \theta + \sqrt{2} = 0;$$

$$\therefore \cos \left( \theta - \frac{\pi}{4} \right) = -1;$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi + \pi;$$

$$\therefore \theta = 2n\pi + \frac{5\pi}{4}, \text{ or } 2n\pi - \frac{3\pi}{4}.$$

9.

$$\operatorname{cosec} \theta + \cot \theta = \sqrt{3};$$

$$\therefore 1 + \cos \theta = \sqrt{3} \sin \theta;$$

$$\therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = -\frac{1}{2};$$

$$\therefore \cos \left( \theta + \frac{\pi}{3} \right) = -\frac{1}{2};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3};$$

$$\theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n-1)\pi;$$

$$\theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n+1)\pi.$$

whence

which may be written

10.

$$\cot \theta - \cot 2\theta = 2;$$

$$\therefore \operatorname{cosec} 2\theta = 2;$$

$$\therefore \theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.$$

11.

$$2 \sin \theta \sin 3\theta = 1;$$

$$\therefore \cos 2\theta - \cos 4\theta = 1;$$

$$\therefore \cos 2\theta = 2 \cos^2 2\theta;$$

$$\therefore \cos 2\theta = 0, \text{ or } \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{4}, \text{ or } n\pi \pm \frac{\pi}{6}.$$

12.

$$\sin 3\theta = 8 \sin^3 \theta;$$

$$\therefore 3 \sin \theta - 4 \sin^3 \theta = 8 \sin^3 \theta;$$

$$\therefore \sin \theta = 4 \sin^3 \theta;$$

$$\therefore \sin \theta = 0, \text{ or } \pm \frac{1}{2};$$

$$\therefore \theta = n\pi, \text{ or } n\pi \pm \frac{\pi}{6}.$$

13.

$$\tan \theta + \tan 3\theta = 2 \tan 2\theta;$$

$$\therefore \frac{\sin 4\theta}{\cos 3\theta \cos \theta} = \frac{2 \sin 2\theta}{\cos 2\theta};$$

$$\therefore \sin 2\theta = 0; \text{ whence } \theta = \frac{n\pi}{2},$$

or

$$\cos^2 2\theta = \cos \theta \cos 3\theta;$$

$$\therefore 2 \cos^2 2\theta = \cos 4\theta + \cos 2\theta = 2 \cos^2 2\theta - 1 + \cos 2\theta;$$

$$\therefore \cos 2\theta = 1;$$

whence

$$\theta = n\pi;$$

$$\therefore \text{all the values are included in } \theta = \frac{n\pi}{2}.$$

14.

$$\cos \theta - \sin \theta = \cos 2\theta;$$

$$\therefore 2 \sin \frac{3\theta}{2} \sin \frac{\theta}{2} = \sin \theta;$$

$$\therefore \sin \frac{\theta}{2} = 0, \text{ whence } \theta = 2n\pi,$$

or

$$\sin \frac{3\theta}{2} = \cos \frac{\theta}{2};$$

$$\therefore \frac{\theta}{2} = 2n\pi + \left( \frac{\pi}{2} - \frac{3\theta}{2} \right);$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, \text{ or } -\theta = 2n\pi - \frac{\pi}{2};$$

$$\therefore \text{the values of } \theta \text{ may be written } 2n\pi, n\pi + \frac{\pi}{4}, 2n\pi + \frac{\pi}{2}.$$

15.

$$\begin{aligned}\operatorname{cosec} \theta + \sec \theta &= 2\sqrt{2}; \\ \therefore \sin \theta + \cos \theta &= 2\sqrt{2} \sin \theta \cos \theta; \\ \therefore \cos \left( \theta - \frac{\pi}{4} \right) &= 2 \sin \theta \cos \theta = \sin 2\theta; \\ \therefore \theta - \frac{\pi}{4} &= 2n\pi \pm \left( \frac{\pi}{2} - 2\theta \right); \\ \theta &= \frac{2n\pi}{3} + \frac{\pi}{4}, \text{ or } 2n\pi + \frac{\pi}{4}.\end{aligned}$$

whence

16.

$$\begin{aligned}\sec \theta - \operatorname{cosec} \theta &= 2\sqrt{2}; \\ \therefore 1 - \sin 2\theta &= 8 \sin^2 \theta \cos^2 \theta = 2 \sin^2 2\theta; \\ \therefore \sin 2\theta &= -1, \text{ or } \frac{1}{2}; \\ \therefore \theta &= n\pi - \frac{\pi}{4}, \text{ or } \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.\end{aligned}$$

The equation may also be solved in the same way as Ex. 15.

17.

$$\begin{aligned}\frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} &= 2; \\ \cos 2\theta - \cos 4\theta &= 2 \cos 4\theta \cos 2\theta \\ &= \cos 6\theta + \cos 2\theta; \\ \therefore \cos 6\theta + \cos 4\theta &= 0, \\ 2 \cos 5\theta \cos \theta &= 0; \\ \therefore \theta &= (2n+1) \frac{\pi}{2}, \text{ or } 5\theta = (2n+1) \frac{\pi}{2}.\end{aligned}$$

18.

$$\begin{aligned}\cos 3\theta + 8 \cos^3 \theta &= 0; \\ \therefore 4 \cos^3 \theta &= \cos \theta; \\ \therefore \cos \theta &= 0, \text{ or } \pm \frac{1}{2}; \\ \therefore \theta &= \frac{(2n+1)\pi}{2}, 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \left( \pi - \frac{\pi}{3} \right); \\ \text{that is, the values of } \theta &\text{ are } \frac{(2n+1)\pi}{2}, n\pi \pm \frac{\pi}{3}.\end{aligned}$$

19.

$$\begin{aligned}1 + \sqrt{3} \tan^2 \theta &= (1 + \sqrt{3}) \tan \theta; \\ \therefore (\sqrt{3} \tan \theta - 1)(\tan \theta - 1) &= 0; \\ \therefore \tan \theta &= 1, \text{ or } \frac{1}{\sqrt{3}}; \\ \therefore \theta &= n\pi + \frac{\pi}{4}, \text{ or } n\pi + \frac{\pi}{6}.\end{aligned}$$

$$\begin{aligned}
 20. \quad & \tan^3 \theta + \cot^3 \theta = 8 \operatorname{cosec}^3 2\theta + 12; \\
 & \therefore \sin^6 \theta + \cos^6 \theta = 1 + 12 \sin^3 \theta \cos^3 \theta; \\
 & \therefore \sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta = 1 + 12 \sin^3 \theta \cos^3 \theta; \\
 & \therefore 1 - 3 \sin^2 \theta \cos^2 \theta = 1 + 12 \sin^3 \theta \cos^3 \theta; \\
 & \therefore \sin^2 \theta \cos^2 \theta (4 \sin \theta \cos \theta + 1) = 0;
 \end{aligned}$$

whence  $\sin 2\theta = 0$ , or  $-\frac{1}{2}$ ;

$$\therefore \theta = \frac{n\pi}{2}, \text{ or } \frac{n\pi}{2} + (-1)^{n+1} \frac{\pi}{12}.$$

$$21. \quad \sin \theta = \sqrt{2} \sin \phi, \quad \sqrt{3} \cos \theta = \sqrt{2} \cos \phi;$$

$$\therefore \text{by squaring and adding we have } \sin^2 \theta + 3 \cos^2 \theta = 2,$$

that is,  $1 + 2 \cos^2 \theta = 2$ , whence  $\cos \theta = \pm \frac{1}{\sqrt{2}}$ ;

$$\therefore \theta = 2n\pi \pm \frac{\pi}{4}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{4}\right);$$

both of which are included in  $\theta = n\pi \pm \frac{\pi}{4}$ .

Again, we have  $\cos \phi = \frac{\sqrt{3}}{\sqrt{2}} \cos \theta = \pm \frac{\sqrt{3}}{2}$ ;

$$\therefore \phi = 2n\pi \pm \frac{\pi}{6}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{6}\right),$$

which are both included in  $\phi = n\pi \pm \frac{\pi}{6}$ .

$$22. \quad \operatorname{cosec} \theta = \sqrt{3} \operatorname{cosec} \phi, \quad \cot \theta = 3 \cot \phi.$$

By squaring and subtracting we have

$$1 = 3 (\operatorname{cosec}^2 \phi - 3 \cot^2 \phi) = 3 (1 - 2 \cot^2 \phi);$$

$$\therefore \cot \phi = \pm \frac{1}{\sqrt{3}}; \text{ whence } \phi = n\pi \pm \frac{\pi}{3}.$$

Also  $\cot \theta = \pm \sqrt{3}$ ; whence  $\theta = n\pi \pm \frac{\pi}{6}$ .

$$23. \quad \sec \phi = \sqrt{2} \sec \theta, \quad \tan \phi = \sqrt{3} \tan \theta.$$

Subtracting the square of the second equation from the square of the first, we obtain

$$1 = 2 \sec^2 \theta - 3 \tan^2 \theta = 2 - \tan^2 \theta;$$

$$\therefore \tan \theta = \pm 1; \text{ whence } \theta = n\pi \pm \frac{\pi}{4}.$$

Also  $\tan \phi = \pm \sqrt{3}$ ; whence  $\phi = n\pi \pm \frac{\pi}{3}$ .

24. If  $\theta + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6};$

then

$$\sin\left(\theta + \frac{\pi}{4}\right) = \sin \frac{\pi}{6};$$

$$\therefore \cos\left(\frac{\pi}{2} - \theta - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right);$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \cos \frac{\pi}{3};$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

Which proves that the same series of angles are given by the two equations.

25. In the figure on page 252 let  $OE$  be drawn bisecting the angle  $P_1OP_3$ , and let the angles  $P_1OE$ ,  $P_2OE$  each be equal to  $a$ ;

then the formula  $\left(2n + \frac{1}{4}\right)\pi \pm a$  comprises angles whose boundary lines are  $OP_1$  or  $OP_3$ .

Again in the formula  $\left(n - \frac{1}{4}\right)\pi + (-1)^n\left(\frac{\pi}{2} - a\right)$  the terms  $n\pi + (-1)^n \frac{\pi}{2}$  comprise angles whose boundary lines are  $OY$ , whether  $n$  be odd or even.

Hence the whole formula comprises angles formed by starting again from  $OY$  and turning through an angle  $-\frac{\pi}{4} - (-1)^n a$ ; that is,  $-\frac{\pi}{4} \pm a$ .

Hence the second formula also comprises angles whose boundary lines are  $OP_1$  or  $OP_3$ .

### EXAMPLES. XIX. c. PAGE 242.

1. Let  $\theta = \sin^{-1} \frac{12}{13}$ ; then  $\operatorname{cosec} \theta = \frac{13}{12}$ .

$$\therefore \cot^2 \theta = \frac{169}{144} - 1 = \frac{25}{144};$$

$$\therefore \theta = \cot^{-1} \frac{5}{12}.$$

2. Let  $\theta = \operatorname{cosec}^{-1} \frac{17}{8}$ ; then  $\operatorname{cosec} \theta = \frac{17}{8}$ , and  $\cot \theta = \frac{15}{8}$ .

$$\therefore \theta = \tan^{-1} \frac{8}{15}.$$

3. Let

$$\theta = \tan^{-1} x; \text{ then } \tan \theta = x;$$

$$\therefore \sec^2 \theta = 1 + \tan^2 \theta = 1 + x^2,$$

$$\sec(\tan^{-1} x) = \sqrt{1 + x^2}.$$

that is,

$$4. \quad 2 \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \tan^{-1} \frac{3}{4}.$$

$$5. \quad \tan^{-1} \frac{4}{3} - \tan^{-1} 1 = \tan^{-1} \frac{\frac{4}{3} - 1}{1 + \frac{4}{3}} = \tan^{-1} \frac{1}{7}.$$

$$6. \quad \tan^{-1} \frac{2}{11} + \cot^{-1} \frac{24}{7} = \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} \\ = \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{14}{264}} = \tan^{-1} \frac{1}{2}.$$

$$7. \quad \cot^{-1} \frac{4}{3} - \cot^{-1} \frac{15}{8} = \cot^{-1} \frac{\frac{4}{3} \cdot \frac{15}{8} + 1}{\frac{15}{8} - \frac{4}{3}} = \cot^{-1} \frac{84}{13}.$$

$$8. \quad 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{25}} + \tan^{-1} \frac{1}{4} \\ = \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{1}{4} \\ = \tan^{-1} \frac{\frac{5}{12} + \frac{1}{4}}{1 - \frac{5}{48}} = \tan^{-1} \frac{32}{43}.$$

$$9. \quad \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \tan^{-1} 1.$$

$$\text{Again,} \quad \tan^{-1} \frac{5}{6} + \tan^{-1} \frac{1}{11} = \tan^{-1} \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{66}} = \tan^{-1} 1.$$

$$10. \quad \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} = \tan^{-1} \frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{56}} + \tan^{-1} \frac{1}{18}$$

$$= \tan^{-1} \frac{3}{11} + \tan^{-1} \frac{1}{18} = \tan^{-1} \frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{198}}$$

$$= \tan^{-1} \frac{1}{3} = \cot^{-1} 3.$$

$$11. \quad \tan^{-1} \frac{3}{5} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{5} + \tan^{-1} \frac{3}{4}$$

$$= \tan^{-1} \frac{\frac{3}{5} + \frac{3}{4}}{1 - \frac{9}{20}} = \tan^{-1} \frac{27}{11}.$$

$$12. \quad 2 \cot^{-1} \frac{5}{4} = 2 \tan^{-1} \frac{4}{5} = \tan^{-1} \frac{\frac{8}{5}}{1 - \frac{16}{25}} = \tan^{-1} \frac{40}{9}.$$

$$13. \quad 2 \tan^{-1} \frac{8}{15} = \tan^{-1} \frac{\frac{16}{15}}{1 - \frac{64}{225}} = \tan^{-1} \frac{240}{161} = \sin^{-1} \frac{240}{\sqrt{161^2 + 240^2}}$$

$$= \sin^{-1} \frac{240}{289}.$$

$$14. \quad \text{Let } \sin^{-1} x = \theta; \text{ then } \sin \theta = x, \cos \theta = \sqrt{1-x^2}.$$

$$\therefore \sin(2 \sin^{-1} x) = \sin 2\theta = 2 \sin \theta \cos \theta = 2x \sqrt{1-x^2}.$$

$$15. \quad \text{Let } \sin^{-1} \sqrt{\frac{1-x}{2}} = \theta; \text{ then } \sin \theta = \sqrt{\frac{1-x}{2}}.$$

$$\text{Now } \cos 2\theta = 1 - (1-x) = x; \text{ whence } 2\theta = \cos^{-1} x.$$

$$\text{That is, } \cos^{-1} x = 2 \sin^{-1} \sqrt{\frac{1-x}{2}}.$$

16. Let  $\tan^{-1} \sqrt{\frac{x}{a}} = \theta$ ; then  $\tan \theta = \sqrt{\frac{x}{a}}$ .

Now  $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{a - x}{a + x}$ ; whence  $2\theta = \cos^{-1} \frac{a - x}{a + x}$ .

That is,  $2 \tan^{-1} \sqrt{\frac{x}{a}} = \cos^{-1} \frac{a - x}{a + x}$ .

17.  $2 \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{\frac{1}{8} + \frac{1}{5}}{1 - \frac{1}{40}} + \tan^{-1} \frac{1}{7}$

$$= 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{28}} = \tan^{-1} 1 = \frac{\pi}{4}.$$

18. Let  $\theta = \sin^{-1} a$ ,  $\phi = \cos^{-1} b$ ;

then  $\sin \theta = a$ ,  $\cos \theta = \sqrt{1 - a^2}$ ,  $\cos \phi = b$ ,  $\sin \phi = \sqrt{1 - b^2}$ ;

$\therefore \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi = b \sqrt{1 - a^2} + a \sqrt{1 - b^2}$ ;

$\therefore \sin^{-1} a - \cos^{-1} b = \theta - \phi = \cos^{-1} \{b \sqrt{1 - a^2} + a \sqrt{1 - b^2}\}.$

[In some of the examples which follow diagrams may be used with advantage as in Examples 2 and 3 of Art. 249.]

19.  $\sin^{-1} \frac{4}{5} + \cos^{-1} \frac{2}{\sqrt{5}} = \cot^{-1} \frac{3}{4} + \cot^{-1} 2 = \cot^{-1} \frac{\frac{3}{4} - 1}{\frac{3}{4} + 2} = \cot^{-1} \frac{2}{11}.$

20.  $\cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} = \cos^{-1} \frac{63}{65} + \tan^{-1} \frac{5}{12}$   
 $= \tan^{-1} \frac{\sqrt{65^2 - 63^2}}{63} + \tan^{-1} \frac{5}{12}$   
 $= \tan^{-1} \frac{16}{63} + \tan^{-1} \frac{5}{12}$   
 $= \tan^{-1} \frac{192 + 315}{756 - 80} = \tan^{-1} \frac{507}{676} = \tan^{-1} \frac{3}{4}$   
 $= \sin^{-1} \frac{3}{5}.$

$$\begin{aligned}
 21. \quad \tan^{-1} m + \tan^{-1} n &= \tan^{-1} \frac{m+n}{1-mn} \\
 &= \cos^{-1} \frac{1-mn}{\sqrt{(m+n)^2 + (1-mn)^2}} = \cos^{-1} \frac{1-mn}{\sqrt{1+m^2+n^2+m^2n^2}} \\
 &= \cos^{-1} \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \text{Let} \quad \cos^{-1} \frac{20}{29} &= \theta, \quad \tan^{-1} \frac{16}{63} = \phi; \\
 \text{then we have} \quad \sin \theta &= \frac{21}{29}, \quad \sin \phi = \frac{16}{65}, \quad \cos \phi = \frac{63}{65}; \\
 \therefore \cos(\theta - \phi) &= \cos \theta \cos \phi + \sin \theta \sin \phi = \frac{20}{29} \cdot \frac{63}{65} + \frac{21}{29} \cdot \frac{16}{65} = \frac{1596}{1885}; \\
 \text{that is,} \quad \cos^{-1} \frac{20}{29} - \tan^{-1} \frac{16}{63} &= \theta - \phi = \cos^{-1} \frac{1596}{1885}.
 \end{aligned}$$

$$23. \quad \text{Let } \cos^{-1} \sqrt{\frac{2}{3}} = \theta, \text{ then } \cos \theta = \sqrt{\frac{2}{3}}, \quad \sin \theta = \frac{1}{\sqrt{3}}.$$

$$\begin{aligned}
 \text{Now} \quad \cos\left(\theta - \frac{\pi}{6}\right) &= \cos \theta \cos \frac{\pi}{6} + \sin \theta \sin \frac{\pi}{6} \\
 &= \sqrt{\frac{2}{3}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \cdot \frac{1}{2} \\
 &= \frac{\sqrt{6} + 1}{2\sqrt{3}}.
 \end{aligned}$$

$$\therefore \theta - \frac{\pi}{6} = \cos^{-1} \frac{\sqrt{6} + 1}{2\sqrt{3}};$$

$$\text{that is,} \quad \cos^{-1} \sqrt{\frac{2}{3}} - \cos^{-1} \frac{\sqrt{6} + 1}{2\sqrt{3}} = \frac{\pi}{6}.$$

$$\begin{aligned}
 24. \quad \text{Second side} &= 2 \cdot \frac{x+x^{-1}}{1-x^4} = \frac{2x}{1-x^2} \\
 &= \tan\left(\tan^{-1} \frac{2x}{1-x^2}\right) \\
 &= \tan(2 \tan^{-1} x).
 \end{aligned}$$

$$25. \quad \text{Second side} = \tan^{-1} a - \tan^{-1} b + \tan^{-1} b - \tan^{-1} c + \tan^{-1} c = \tan^{-1} a.$$

$$\begin{aligned}
 26. \quad \text{Let} \quad \tan^{-1} x &= \alpha, \quad \tan^{-1} y = \beta, \quad \tan^{-1} z = \gamma; \\
 \text{then} \quad \alpha + \beta + \gamma &= \pi; \\
 \therefore \tan \alpha + \tan \beta + \tan \gamma &= \tan \alpha \tan \beta \tan \gamma; \quad [\text{Art. 135, Ex. 2.}] \\
 \text{that is,} \quad x + y + z &= xyz.
 \end{aligned}$$

27. We have  $u = \cot^{-1} \sqrt{\cos a} - \cot^{-1} \sqrt{\frac{1}{\cos a}}$

$$= \cot^{-1} \frac{1+1}{\sqrt{\frac{1}{\cos a}} - \sqrt{\cos a}} = \cot^{-1} \frac{2\sqrt{\cos a}}{1-\cos a};$$

$$\therefore \cot^2 u = \frac{4 \cos a}{(1-\cos a)^2};$$

$$\therefore \operatorname{cosec}^2 u = 1 + \cot^2 u = \left( \frac{1+\cos a}{1-\cos a} \right)^2;$$

$$\therefore \sin u = \frac{1-\cos a}{1+\cos a} = \tan^2 \frac{a}{2}.$$

## EXAMPLES. XIX. d. PAGE 244.

1.  $\sin^{-1} x = \cos^{-1} x = \sin^{-1} \sqrt{1-x^2};$   
 $\therefore x = \sqrt{1-x^2};$  whence  $x = \pm \frac{1}{\sqrt{2}}.$

2.  $\tan^{-1} x = \cot^{-1} x = \tan^{-1} \frac{1}{x};$   
 $\therefore x = \frac{1}{x};$  whence  $x = \pm 1.$

3.  $\tan^{-1} (x+1) - \tan^{-1} (x-1) = \cot^{-1} 2;$   
 $\therefore \tan^{-1} \frac{2}{x^2} = \tan^{-1} \frac{1}{2};$  whence  $x = \pm 2.$

4.  $\cot^{-1} x + \cot^{-1} 2x = \frac{3\pi}{4};$   
 $\therefore \cot^{-1} \frac{2x^2-1}{3x} = \frac{3\pi}{4};$   
 $\therefore \frac{2x^2-1}{3x} = \cot \frac{3\pi}{4} = -1;$   
 $\therefore 2x^2+3x-1=0;$  whence  $x = \frac{-3 \pm \sqrt{17}}{4}.$

5.  $\sin^{-1} x - \cos^{-1} x = \sin^{-1} (3x-2);$   
 $\therefore \sin (\sin^{-1} x - \cos^{-1} x) = 3x-2;$   
 $\therefore x^2 - \sqrt{1-x^2} \cdot \sqrt{1-x^2} = 3x-2;$

that is,

$$2x^2 - 1 = 3x - 2;$$

$$\therefore 2x^2 - 3x + 1 = 0; \text{ whence } x = 1, \text{ or } \frac{1}{2}.$$

6.

$$\begin{aligned}\cos^{-1} x - \sin^{-1} x &= \cos^{-1} x \sqrt{3}; \\ \therefore \cos(\cos^{-1} x - \sin^{-1} x) &= x \sqrt{3}; \\ \therefore x \sqrt{1-x^2} + x \sqrt{1-x^2} &= x \sqrt{3};\end{aligned}$$

that is,

$$2x \sqrt{1-x^2} = x \sqrt{3}; \text{ whence } x=0, \text{ or } \pm \frac{1}{2}.$$

7.

$$\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4};$$

$$\therefore \tan^{-1} \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2} \cdot \frac{x+1}{x+2}} = \frac{\pi}{4};$$

$$\therefore \frac{2x^2-4}{-3} = 1;$$

$$\therefore 2x^2 = 1, \text{ or } x = \pm \frac{1}{\sqrt{2}}.$$

8.

$$2 \cot^{-1} 2 + \cos^{-1} \frac{3}{5} = \operatorname{cosec}^{-1} x;$$

$$\therefore \cot^{-1} \frac{3}{4} + \cot^{-1} \frac{3}{4} = \operatorname{cosec}^{-1} x;$$

$$\therefore \cot^{-1} \frac{\frac{9}{16} - 1}{\frac{3}{2}} = \operatorname{cosec}^{-1} x;$$

$$\therefore \cot^{-1} -\frac{7}{24} = \operatorname{cosec}^{-1} x = \cot^{-1} \sqrt{x^2-1};$$

$$\therefore \frac{49}{24^2} = x^2 - 1; \text{ whence } x = \pm \frac{25}{24}.$$

9.

$$\tan^{-1} x + \tan^{-1} (1-x) = 2 \tan^{-1} \sqrt{x-x^2};$$

$$\therefore \tan^{-1} \frac{1}{1-x+x^2} = \tan^{-1} \frac{2\sqrt{x-x^2}}{1-x+x^2};$$

$$\therefore 1 = 4(x-x^2); \text{ whence } x = \frac{1}{2}.$$

10.

$$\cos^{-1} \frac{1-a^2}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2} = 2 \tan^{-1} x;$$

$$\therefore \tan^{-1} \frac{2a}{1-a^2} - \tan^{-1} \frac{2b}{1-b^2} = 2 \tan^{-1} x;$$

$$\therefore 2 \tan^{-1} a - 2 \tan^{-1} b = 2 \tan^{-1} x;$$

$$\therefore \tan^{-1} a - \tan^{-1} b = \tan^{-1} x; \text{ whence } x = \frac{a-b}{1+ab}.$$

$$11. \quad \sin^{-1} \frac{2a}{1+a^2} + \tan^{-1} \frac{2x}{1-x^2} = \cos^{-1} \frac{1-b^2}{1+b^2};$$

$$\therefore \tan^{-1} \frac{2a}{1-a^2} + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{2b}{1-b^2};$$

$$\therefore 2 \tan^{-1} a + 2 \tan^{-1} x = 2 \tan^{-1} b; \text{ whence } x = \frac{b-a}{1+ab}.$$

$$12. \quad \cot^{-1} \frac{x^2-1}{2x} + \tan^{-1} \frac{2x}{x^2-1} + \frac{4\pi}{3} = 0;$$

$$\therefore 2 \tan^{-1} \frac{2x}{x^2-1} + \frac{4\pi}{3} = 0;$$

$$\therefore \tan^{-1} \frac{2x}{1-x^2} = \frac{2\pi}{3};$$

$$\tan^{-1} x = \frac{\pi}{3}; \text{ whence } x = \sqrt{3}.$$

$$13. \quad \sin^{-1} \frac{2ab}{a^2+b^2} = \sin^{-1} \frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}} = \tan^{-1} \frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}};$$

$$\begin{aligned} \therefore \sin^{-1} \frac{2ab}{a^2+b^2} + \sin^{-1} \frac{2cd}{c^2+d^2} &= 2 \tan^{-1} \frac{b}{a} + 2 \tan^{-1} \frac{d}{c} \\ &= 2 \tan^{-1} \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{bd}{ac}} \\ &= 2 \tan^{-1} \frac{bc+ad}{ac-bd} \\ &= 2 \tan^{-1} \frac{y}{x} = \sin^{-1} \frac{2xy}{x^2+y^2}, \end{aligned}$$

where

$$y = bc + ad, \quad x = ac - bd.$$

$$14. \quad \sin [2 \cos^{-1} \{ \cot (2 \tan^{-1} x) \}] = 0;$$

$$\therefore \sin \left[ 2 \cos^{-1} \left\{ \cot \left( \tan^{-1} \frac{2x}{1-x^2} \right) \right\} \right] = 0;$$

$$\therefore \sin \left[ 2 \cos^{-1} \frac{1-x^2}{2x} \right] = 0;$$

$$\therefore \frac{1-x^2}{2x} \cdot \sqrt{1 - \left( \frac{1-x^2}{2x} \right)^2} = 0;$$

whence  
that is,

$$x = \pm 1, \text{ or } 1-x^2 = \pm 2x;$$

$$x = \pm 1, \text{ or } \pm(1 \pm \sqrt{2}).$$

15.

$$2 \tan^{-1} (\cos \theta) = \tan^{-1} (2 \operatorname{cosec} \theta);$$

$$\therefore \frac{2 \cos \theta}{1 - \cos^2 \theta} = 2 \operatorname{cosec} \theta;$$

$$\therefore \frac{2 \cos \theta}{\sin^2 \theta} = \frac{2}{\sin \theta};$$

$$\therefore \cot \theta = 1; \text{ whence } \theta = n\pi + \frac{\pi}{4}.$$

16.

$$\sin (\pi \cos \theta) = \cos (\pi \sin \theta);$$

$$\therefore \pi \sin \theta = \frac{\pi}{2} \pm \pi \cos \theta;$$

$$\therefore \sin \theta \pm \cos \theta = \frac{1}{2};$$

$$\therefore 1 \pm 2 \sin \theta \cos \theta = \frac{1}{4};$$

$$\therefore \sin 2\theta = \pm \frac{3}{4};$$

$$\therefore 2\theta = \pm \sin^{-1} \frac{3}{4}.$$

17.

$$\sin (\pi \cot \theta) = \cos (\pi \tan \theta);$$

$$\therefore \pi \tan \theta = 2n\pi + \left( \frac{\pi}{2} - \pi \cot \theta \right);$$

$$\therefore \tan \theta \pm \cot \theta = \frac{4n \pm 1}{2};$$

$$\therefore \frac{\sin^2 \theta \pm \cos^2 \theta}{\sin 2\theta} = \frac{4n \pm 1}{4};$$

that is, either  $\cot 2\theta$  or  $\operatorname{cosec} 2\theta$  is of the form  $\frac{4n+1}{4}$ .

18.

$$\tan (\pi \cot \theta) = \cot (\pi \tan \theta);$$

$$\therefore \pi \tan \theta = n\pi + \frac{\pi}{2} - \pi \cot \theta;$$

$$\therefore \tan^2 \theta - \frac{2n+1}{2} \tan \theta + 1 = 0;$$

$$\therefore \tan \theta = \frac{2n+1 \pm \sqrt{(2n+1)^2 - 4}}{4}$$

$$= \frac{2n+1}{4} \pm \frac{\sqrt{4n^2 + 4n - 15}}{4}.$$

19.

$$\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3;$$

$$\therefore \frac{x + \frac{1}{y}}{1 - \frac{x}{y}} = 3;$$

$$\therefore y = \frac{3x+1}{3-x};$$

thus we see that the only positive integral values which  $x$  may have are 1, 2;  
when

$$x=1, y=2;$$

when

$$x=2, y=7;$$

and these are all the positive integral solutions of the equation.

### MISCELLANEOUS EXAMPLES. G. PAGE 246.

1. Since

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

we have

$$\frac{a}{4} = \frac{b}{5} = \frac{c}{6};$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{45}{60} = \frac{3}{4}.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{27}{48} = \frac{9}{16}.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5}{40} = \frac{1}{8}.$$

That is, the cosines are in the ratio of 12 : 9 : 2.

2. (1)

$$2 \cos^3 \theta + \sin^2 \theta - 1 = 0;$$

$$\therefore 2 \cos^3 \theta = 1 - \sin^2 \theta = \cos^2 \theta;$$

$$\therefore \cos \theta = 0, \text{ or } \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, \text{ or } 2n\pi \pm \frac{\pi}{3}.$$

(2)

$$\sec^3 \theta - 2 \tan^2 \theta = 2;$$

$$\therefore \sec^3 \theta = 2(1 + \tan^2 \theta) = 2 \sec^2 \theta;$$

$$\therefore \sec \theta = 2, \text{ since } \sec \theta = 0 \text{ is inadmissible};$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

3.

$$\tan \beta = 2 \sin \alpha \sin \gamma \operatorname{cosec} (\alpha + \gamma);$$

$$\therefore 2 \cot \beta = \frac{\sin (\alpha + \gamma)}{\sin \alpha \sin \gamma} = \cot \alpha + \cot \gamma;$$

$\therefore \cot \alpha, \cot \beta, \cot \gamma$  are in arithmetical progression.

$$\begin{aligned}
 4. \quad 4r(r_1 + r_2 + r_3) &= \frac{4\Delta^2}{s} \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) \\
 &= 4 \{ (s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b) \} \\
 &= 4 \{ 3s^2 - 2(a+b+c)s + bc + ca + ab \} \\
 &= 4(bc + ca + ab) - 4s^2 \\
 &= 4(bc + ca + ab) - (a+b+c)^2 \\
 &= 2(bc + ca + ab) - (a^2 + b^2 + c^2).
 \end{aligned}$$

$$\begin{aligned}
 5. (1) \quad \tan^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} &= \tan^{-1} \frac{\frac{1}{3} - \frac{1}{5}}{1 + \frac{1}{15}} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{\frac{1}{8} + \frac{1}{7}}{1 - \frac{1}{56}} = \tan^{-1} \frac{3}{11}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \cos \left( \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} \right) &= \frac{4}{5} \cdot \frac{15}{17} - \frac{3}{5} \cdot \frac{8}{17} \\
 &= \frac{36}{85};
 \end{aligned}$$

$$\begin{aligned}
 \therefore \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} &= \cos^{-1} \frac{36}{85} \\
 &= \frac{\pi}{2} - \sin^{-1} \frac{36}{85}.
 \end{aligned}$$

$$6. \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{333 \times 74}{185 \times 222}} = \sqrt{\frac{6}{10}}.$$

$$\begin{aligned}
 \log \cos \frac{C}{2} &= \frac{1}{2} (\log 6 - 1) = \frac{1}{2} (\bar{1}.7781513) \\
 &= \bar{1}.8890756;
 \end{aligned}$$

$$\begin{array}{r}
 \log \cos 39^\circ 14' = \bar{1}.8890644 \\
 \text{diff.} \qquad \qquad \qquad 112
 \end{array}$$

$$\text{prop}^l. \text{ decrease} = \frac{112}{1032} \times 60'' = 6.5'',$$

$$\therefore \frac{C}{2} = 39^\circ 13' 53.5'', \text{ and } C = 78^\circ 27' 47''.$$

$$7. \quad \tan(\alpha + \theta) = n \tan(\alpha - \theta);$$

$$\therefore \frac{\tan(\alpha + \theta)}{\tan(\alpha - \theta)} = n;$$

$$\therefore \frac{\tan(\alpha + \theta) - \tan(\alpha - \theta)}{\tan(\alpha + \theta) + \tan(\alpha - \theta)} = \frac{n - 1}{n + 1};$$

that is,

$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{n - 1}{n + 1}.$$

$$8. \quad 8R^2 = a^2 + b^2 + c^2;$$

$$\therefore 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$= 2 - \cos(A + B) \cos(A - B) - \cos^2 C$$

$$= 2 + 2 \cos A \cos B \cos C;$$

$$\therefore \cos A \cos B \cos C = 0;$$

that is, one of the angles of the triangle is a right angle.

$$9. \quad \text{Area of inscribed polygon} = nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n};$$

$$\text{Area of circumscribed polygon} = nr^2 \tan \frac{\pi}{n};$$

$\therefore$  we have

$$\cos^2 \frac{\pi}{n} = \frac{3}{4};$$

$$\therefore \cos \frac{\pi}{n} = \frac{\sqrt{3}}{2};$$

$$\therefore n = 6.$$

10. Let  $P, Q$  be the positions of the boats; then we have

$$\angle PBA = \angle QBC = 45^\circ, \quad \angle PCB = 15^\circ, \quad \angle QCD = 75^\circ.$$

$$\therefore PB = \frac{400 \sin 15^\circ}{\sin 30^\circ} = 200 \sqrt{2} (\sqrt{3} - 1);$$

$$QB = \frac{400 \sin 75^\circ}{\sin 30^\circ} = 200 \sqrt{2} (\sqrt{3} + 1).$$

But  $PBQ$  is a right angle;

$$\therefore PQ^2 = PB^2 + QB^2 = 200^2 (12 + 4) = 200^2 \times 16;$$

$$\therefore PQ = 800.$$

$$\begin{aligned} \text{Again, the distance of } P \text{ from } AB &= PB \sin 45^\circ = 200 (\sqrt{3} - 1) \\ &= 146.4 \text{ yds.,} \end{aligned}$$

$$\begin{aligned} \text{and the distance of } Q \text{ from } AB &= QB \sin 75^\circ = 200 (\sqrt{3} + 1) \\ &= 546.4 \text{ yds.} \end{aligned}$$

**EXAMPLES. XX. a.** PAGE 255.

1. When  $\frac{A}{2}$  lies between  $-135^\circ$  and  $-180^\circ$ ,  $\sin \frac{A}{2}$  is negative, therefore in the first formula of Art. 254, the negative sign must be taken.

2.  $\frac{A}{2}$  lies between  $135^\circ$  and  $180^\circ$ ; and therefore

$\sin \frac{A}{2}$  is positive,  $\cos \frac{A}{2}$  is negative.

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \frac{119}{169}}{2}} = \frac{5}{13},$$

$$\cos \frac{A}{2} = -\sqrt{\frac{1 + \cos A}{2}} = -\sqrt{\frac{1 + \frac{119}{169}}{2}} = -\frac{12}{13}.$$

3. Here  $\sin \frac{A}{2}$  is negative, and  $\cos \frac{A}{2}$  is positive;

$$\therefore \sin \frac{A}{2} = -\sqrt{\frac{1 - \cos A}{2}} = -\sqrt{\frac{1 - \frac{161}{289}}{2}} = -\frac{15}{17},$$

$$\cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{1 + \frac{161}{289}}{2}} = \frac{8}{17}.$$

4. When  $\frac{A}{2}$  lies between  $135^\circ$  and  $225^\circ$ ,  $\cos \frac{A}{2} > \sin \frac{A}{2}$  and is negative;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A},$$

and  $\sin \frac{A}{2} - \cos \frac{A}{2} = +\sqrt{1 + \sin A}.$

7. When  $\frac{A}{2}$  lies between  $45^\circ$  and  $90^\circ$ ,  $\sin \frac{A}{2} > \cos \frac{A}{2}$  and is positive;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A} = \sqrt{1 + \frac{24}{25}} = \frac{7}{5},$$

and  $\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} = \sqrt{1 - \frac{24}{25}} = \frac{1}{5};$

$$\therefore \sin \frac{A}{2} = \frac{4}{5}, \cos \frac{A}{2} = \frac{3}{5}.$$

8. When  $\frac{A}{2}$  lies between  $135^\circ$  and  $180^\circ$ ,  $\cos \frac{A}{2} > \sin \frac{A}{2}$  and is negative;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A} = -\sqrt{1 - \frac{240}{289}} = -\frac{7}{17},$$

and  $\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} = \sqrt{1 + \frac{240}{289}} = \frac{23}{17};$

$$\therefore \sin \frac{A}{2} = \frac{8}{17}, \cos \frac{A}{2} = -\frac{15}{17}.$$

9. (1) We have

$$\sin A + \cos A = \sqrt{1 + \sin 2A} \dots \dots \dots (i),$$

$$\sin A - \cos A = -\sqrt{1 - \sin 2A} \dots \dots \dots (ii).$$

From (i) we see that of  $\sin A$  and  $\cos A$  the numerically greater is positive.

From (ii) we see that  $\cos A$  is the greater.

Now  $\cos A$  is greater than  $\sin A$  and positive between the limits  $2n\pi - \frac{\pi}{4}$  and  $2n\pi + \frac{\pi}{4}$ .

(3) We have

$$\sin A + \cos A = -\sqrt{1 + \sin 2A} \dots \dots \dots (i),$$

$$\sin A - \cos A = +\sqrt{1 - \sin 2A} \dots \dots \dots (ii).$$

From (i) we see that of  $\sin A$  and  $\cos A$  the numerically greater is negative.

From (ii) we see that  $\cos A$  is the greater.

Now  $\cos A$  is greater than  $\sin A$  and negative between the limits  $2n\pi + \frac{3\pi}{4}$  and  $2n\pi + \frac{5\pi}{4}$ .

10. If  $\frac{A}{2} = 120^\circ$ ,  $\sin \frac{A}{2} > \cos \frac{A}{2}$  and is positive,

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A},$$

$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A}.$$

$$\therefore 2 \sin \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}.$$

$$11. \quad \tan 7\frac{1}{2}^\circ = \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \operatorname{cosec} 15^\circ - \cot 15^\circ$$

$$= \frac{2\sqrt{2}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} - (2 + \sqrt{3})$$

$$= \sqrt{6} + \sqrt{2} - 2 - \sqrt{3};$$

$$\cot 142\frac{1}{2}^\circ = \frac{1 + \cos 285^\circ}{\sin 285^\circ} = \operatorname{cosec} 285^\circ + \cot 285^\circ$$

$$= -\operatorname{cosec} 75^\circ - \cot 75^\circ = -\frac{2\sqrt{2}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} - (2 - \sqrt{3})$$

$$= -\sqrt{2}(\sqrt{3}-1) - 2 + \sqrt{3} = \sqrt{2} + \sqrt{3} - 2 - \sqrt{6}.$$

12.

$$\sin 9^\circ = \frac{1}{4} \{ \sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}} \}$$

[Art. 260.]

$$= \frac{1}{4} \{ \sqrt{5 \cdot 236080} - \sqrt{2 \cdot 7639320} \}$$

$$= \frac{1}{4} \{ 2 \cdot 288 \dots - 1 \cdot 662 \dots \}$$

$$= \frac{.626 \dots}{4} = .156 \dots$$

$$13. \quad (1) \text{ As in Art. 251, } 2 \cos \frac{\pi}{8} = \sqrt{2} + \sqrt{2};$$

but

$$4 \sin^2 \frac{\pi}{16} = 2 - 2 \cos \frac{\pi}{8} = 2 - \sqrt{2} + \sqrt{2};$$

$$\therefore 2 \sin \frac{\pi}{16} = \sqrt{2 - \sqrt{2} + \sqrt{2}}.$$

$$(2) \quad \tan 11^\circ 15' = \frac{1 - \cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \operatorname{cosec} 22\frac{1}{2}^\circ - \cot 22\frac{1}{2}^\circ$$

$$= \frac{2}{\sqrt{2} - \sqrt{2}} - \frac{1}{\sqrt{2} - 1} = \frac{2\sqrt{2} + \sqrt{2}}{\sqrt{2}} - (\sqrt{2} + 1)$$

$$= \sqrt{4 + 2\sqrt{2}} - (\sqrt{2} + 1).$$

$$14. \quad (1) \quad \cos \theta + \sin \theta = \sqrt{2} \cos \left( \theta - \frac{\pi}{4} \right).$$

As  $\theta$  increases from 0 to  $\frac{\pi}{4}$ , the expression is positive and increases from

1 to  $\sqrt{2}$ .

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As  $\theta$  increases from  $\frac{\pi}{4}$  to  $\frac{3\pi}{4}$ , the expression is positive and decreases from  $\sqrt{2}$  to 0.

As  $\theta$  increases from  $\frac{3\pi}{4}$  to  $\frac{5\pi}{4}$ , the expression is negative and increases numerically from 0 to  $-\sqrt{2}$ .

As  $\theta$  increases from  $\frac{5\pi}{4}$  to  $\frac{7\pi}{4}$ , the expression is negative and decreases numerically from  $-\sqrt{2}$  to 0.

As  $\theta$  increases from  $\frac{7\pi}{4}$  to  $2\pi$ , the expression is positive and increases from 0 to 1.

$$(2) \sin \theta - \sqrt{3} \cos \theta = 2 \left( \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta \right) = 2 \sin \left( \theta - \frac{\pi}{3} \right).$$

From  $\theta = 0$  to  $\frac{\pi}{3}$ , the expression is negative and decreases numerically from  $-\sqrt{3}$  to 0.

From  $\theta = \frac{\pi}{3}$  to  $\frac{\pi}{2} + \frac{\pi}{3}$ , or  $\frac{5\pi}{6}$ , the expression is positive and increases from 0 to 2.

From  $\theta = \frac{5\pi}{6}$  to  $\frac{\pi}{2} + \frac{5\pi}{6}$ , or  $\frac{4\pi}{3}$ , the expression is positive and decreases from 2 to 0.

From  $\theta = \frac{4\pi}{3}$  to  $\frac{\pi}{2} + \frac{4\pi}{3}$ , or  $\frac{11\pi}{6}$ , the expression is negative and increases numerically from 0 to  $-2$ .

From  $\theta = \frac{11\pi}{6}$  to  $2\pi$ , the expression is negative and decreases numerically from  $-2$  to  $-\sqrt{3}$ .

$$15. (1) \text{ The expression } = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} = -\frac{1}{\cos 2\theta} = -\sec 2\theta \\ = -\sec \phi, \text{ where } \phi = 2\theta.$$

We have therefore to trace the changes in  $-\sec \phi$ , from  $\phi = 0$  to  $2\pi$ .

From 0 to  $\frac{\pi}{2}$ , the expression is negative and increases numerically from  $-1$  to  $-\infty$ .

From  $\frac{\pi}{2}$  to  $\pi$ , the expression is positive and decreases from  $\infty$  to 1.

From  $\pi$  to  $\frac{3\pi}{2}$ , the expression is positive and increases from 1 to  $\infty$ .

From  $\frac{3\pi}{2}$  to  $2\pi$ , the expression is negative and decreases numerically from  $-\infty$  to  $-1$ .

$$(2) \text{ The expression } = \frac{2 \sin \theta (1 - \cos \theta)}{2 \sin \theta (1 + \cos \theta)} = \tan^2 \frac{\theta}{2}.$$

Now  $\frac{\theta}{2}$  varies from  $0$  to  $\frac{\pi}{2}$ , so that the expression is positive and increases from  $0$  to  $\infty$ .

### EXAMPLES. XX. b. PAGE 260.

1. Here  $\tan \frac{A}{2}$  is negative; hence in the formula  $\frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}$ , the numerator and denominator must have different signs. But when  $A = 320^\circ$ ,  $\tan A$  is negative; therefore we must take the sign which will make the numerator positive,

$$\therefore \tan \frac{A}{2} = \frac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$$

3.  $\tan A = \pm \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$ , and since  $\tan A$  is positive the radical must have the positive sign prefixed.

$$\therefore \tan A = \sqrt{\frac{13 - 12}{13 + 12}} = \frac{1}{5}.$$

4.  $\cot \frac{A}{2} = + \sqrt{\frac{1 + \cos A}{1 - \cos A}}$ ; and when  $\frac{A}{2}$  lies between  $90^\circ$  and  $135^\circ$ ,  $\cot \frac{A}{2}$  is negative;

$$\therefore \cot \frac{A}{2} = - \sqrt{\frac{5 - 4}{5 + 4}} = -\frac{1}{3}.$$

5. The general solution of  $\cot 2\theta = \cot 2a$  is

$$2\theta = n\pi + 2a; \text{ therefore } \theta = \frac{1}{2}(n\pi + 2a).$$

(1) when  $n = 2m$ ,  $\cot \theta = \cot (m\pi + a) = \cot a$ ,

(2) when  $n = 2m + 1$ ,  $\cot \theta = \cot \left( m\pi + \frac{\pi}{2} + a \right) = -\tan a$ .

6.  $\sin \theta = \sin a$ ;  $\therefore \theta = n\pi + (-1)^n a$ ; hence in finding  $\sin \frac{\theta}{3}$  we have to

consider all the values of  $\sin \left( \frac{n\pi}{3} + (-1)^n \frac{a}{3} \right)$ . Give to  $n$  in succession the values  $0, 1, 2, 3, \dots$ . Proceeding as in Art. 264 we shall find that the first six angles are

$$\frac{a}{3}, \frac{\pi}{3} - \frac{a}{3}, \frac{2\pi}{3} + \frac{a}{3}, \pi - \frac{a}{3}, \frac{4\pi}{3} + \frac{a}{3}, \frac{5\pi}{3} - \frac{a}{3},$$

and that the other angles are coterminal with one of these.

Now

$$\sin \frac{2\pi}{3} + \frac{a}{3} = \sin \left\{ \pi - \left( \frac{\pi}{3} - \frac{a}{3} \right) \right\} = \sin \frac{\pi - a}{3};$$

$$\sin \left( \pi - \frac{a}{3} \right) = \sin \frac{a}{3};$$

$$\sin \left( \frac{4\pi}{3} + \frac{a}{3} \right) = \sin \left\{ \pi + \left( \frac{\pi}{3} + \frac{a}{3} \right) \right\} = -\sin \frac{\pi + a}{3};$$

$$\sin \left( \frac{5\pi}{3} - \frac{a}{3} \right) = \sin \left\{ 2\pi - \left( \frac{\pi}{3} + \frac{a}{3} \right) \right\} = -\sin \frac{\pi + a}{3}.$$

Thus the values of  $\sin \frac{\theta}{3}$  are  $\sin \frac{a}{3}$ ,  $\sin \frac{\pi - a}{3}$ ,  $-\sin \frac{\pi + a}{3}$ .

7. Here  $\theta = n\pi + a$ , and we have to find all the values of  $\tan \left( \frac{n\pi}{3} + \frac{a}{3} \right)$ .

Give to  $n$  in succession the values 0, 1, 2, 3, ...; then we shall find that all the angles are coterminal with one of the following:

$$\frac{a}{3}, \quad \frac{\pi}{3} + \frac{a}{3}, \quad \frac{2\pi}{3} + \frac{a}{3}, \quad \pi + \frac{a}{3}, \quad \frac{4\pi}{3} + \frac{a}{3}, \quad \frac{5\pi}{3} + \frac{a}{3},$$

and as in Ex. 6 it may be shewn that the tangents of these angles assume one of the three forms

$$\tan \frac{a}{3}, \quad \tan \frac{\pi + a}{3}, \quad -\tan \frac{\pi - a}{3}.$$

8. Here  $3\theta = 2n\pi \pm 3a$ , or  $\theta = \frac{2n\pi}{3} \pm a$ , and we have to find all the values of  $\sin \left( \frac{2n\pi}{3} \pm a \right)$ .

Now  $n$  must be of the form  $3m$ , or  $3m + 1$ , or  $3m - 1$ .

If  $n = 3m$ ,  $\sin \left( \frac{2n\pi}{3} \pm a \right) = \sin (2m\pi \pm a) = \pm \sin a$ ;

if  $n = 3m + 1$ ,  $\sin \left( \frac{2n\pi}{3} \pm a \right) = \sin \left( 2m\pi + \frac{2\pi}{3} \pm a \right) = \sin \left( \frac{2\pi}{3} \pm a \right)$ ;

if  $n = 3m - 1$ ,  $\sin \left( \frac{2n\pi}{3} \pm a \right) = \sin \left( 2m\pi - \frac{\pi}{3} \pm a \right) = -\sin \left( \frac{\pi}{3} \pm a \right)$ .

9. Here  $3\theta = n\pi + (-1)^n 3a$ , or  $\theta = \frac{n\pi}{3} + (-1)^n a$ , and we have to find all the values of  $\cos \left\{ \frac{n\pi}{3} + (-1)^n a \right\}$ .

If  $n = 3m$ ,  $\cos \left\{ \frac{n\pi}{3} + (-1)^n a \right\} = \cos \{ m\pi + (-1)^{3m} a \} = \pm \cos a$ ;

$$\text{if } n=3m+1, \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} = \cos \left\{ m\pi + \frac{\pi}{3} + (-1)^{3m+1} \alpha \right\} \\ = \pm \cos \left( \frac{\pi}{3} \pm \alpha \right);$$

$$\text{if } n=3m+2, \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} = \cos \left\{ m\pi + \frac{2\pi}{3} + (-1)^{3m+2} \alpha \right\} \\ = \pm \cos \left( \frac{2\pi}{3} \pm \alpha \right).$$

### EXAMPLES. XXI. a. PAGE 267.

1. Let  $x$  = distance in feet, then  $x = 44 \cot 35' = \frac{44}{\theta}$ , nearly, where  $\theta$  is the radian measure of  $35'$ .

$$\text{That is, } x = 44 \times \frac{60}{35} \times \frac{180}{\pi} = \frac{44 \times 6 \times 180}{11} \text{ ft.};$$

whence  $x = 1440$  yds.

$$2. \text{ Here } x = \frac{22}{3} \cot 24' 30'' = \frac{22}{3} \times \frac{180 \times 60}{77} \text{ ft., nearly} \\ = 342\frac{6}{7} \text{ yds.}$$

3. With the figure of Ex. 1, p. 264, we have

$$PN = 840 \tan 1^\circ 30' = 840 \times \frac{3}{2} \times \frac{\pi}{180} \text{ yds., nearly} \\ = 840 \times \frac{11}{420} = 22 \text{ yds.}$$

4. Let  $\theta$  be the required angle, then approximately

$$\theta = \tan \theta = \frac{121}{1760 \times 3 \times 12} = \frac{11}{160 \times 3 \times 12} \text{ radians} \\ = \frac{11}{160 \times 3 \times 12} \times \frac{180 \times 7}{22} \times 60 \text{ minutes} \\ = 6' 34''.$$

5. Let  $\theta$  be the required angle, then approximately

$$\frac{\theta}{2} = \tan \frac{\theta}{2} = \frac{2}{3000} = \frac{1}{1500} \text{ radians;} \\ \therefore \theta = \frac{1}{750} \times \frac{180 \times 7}{22} \times 60 \text{ minutes} = 4' 35''.$$

6. The radian measure of  $\frac{1^\circ}{4} = \frac{22}{4 \times 180 \times 7} = \frac{11}{14 \times 180}$ ;

$$\therefore x = .625 \cot \frac{1^\circ}{4} = \frac{14 \times 180}{11} \times .625 \text{ inches, nearly}$$

$$= 11 \text{ ft. } 11 \text{ in.}$$

7. The radian measure of  $5' = \frac{5 \times 22}{60 \times 180 \times 7} = \frac{11}{42 \times 180}$ ;

$$\therefore x = \frac{11}{2} \cot \frac{5'}{2} = \frac{11}{2} \times \frac{84 \times 180}{11} \text{ inches, nearly}$$

$$= 210 \text{ yards.}$$

8. Let  $\theta$  be the difference between the latitudes, then

$$\theta = \frac{11}{3960} = \frac{1}{360} \text{ radians}$$

$$= \frac{180 \times 7 \times 60}{360 \times 22} \text{ minutes} = 9' 33''.$$

9. See figure of Example 2, page 264.

Let  $DC$  be the man,  $CB$  the tower,  $A$  the point distant 24 feet from the tower;

Let  $\angle BAC = \alpha$ ,  $\angle CAD = \theta$ ;

then  $\tan \alpha = \frac{120}{24} = 5,$

$$\tan (\alpha + \theta) = \frac{126}{24} = \frac{21}{4}.$$

But  $\tan (\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} = \frac{\tan \alpha + \theta}{1 - \theta \tan \alpha}$ , approximately;

$$\therefore \frac{21}{4} = \frac{5 + \theta}{1 - 5\theta};$$

whence  $\theta = \frac{1}{109} \text{ radians} = 31.5', \text{ nearly.}$

10. See figure of Example 2, page 264.

Let  $DC$  be the flagstaff,  $CB$  the cliff, and let  $DC = x$  feet; then taking the angles as before, we have

$$\tan \alpha = \frac{490}{980} = \frac{1}{2} = .5, \quad \tan \theta = .04 \text{ approximately;}$$

$$\therefore \tan (\alpha + \theta) = \frac{.5 + .04}{1 - .02} = \frac{54}{98} = \frac{27}{49};$$

$$\therefore \frac{x + 490}{980} = \frac{27}{49};$$

whence  $x = 50;$

that is, the height of the flagstaff is 50 feet.

$$11. (1) n' = \frac{n\pi}{180 \times 60} \text{ radians};$$

$$\therefore \text{Lt.}_{n=0} \left( \frac{\sin n'}{n} \right) = \frac{n\pi}{180 \times 60} \times \frac{1}{n} = \frac{\pi}{10800}.$$

$$(2) n'' = \frac{n\pi}{180 \times 60 \times 60} \text{ radians};$$

$$\therefore \text{Lt.}_{n=0} \left( \frac{\sin n''}{n} \right) = \frac{n\pi}{180 \times 60 \times 60} \times \frac{1}{n} = \frac{\pi}{648000}.$$

$$12. \frac{1}{2} nr^2 \sin \frac{2\pi}{n} = \pi r^2 \cdot \frac{n}{2\pi} \cdot \sin \frac{2\pi}{n} = \pi r^2 \left( \sin \frac{2\pi}{n} \div \frac{2\pi}{n} \right);$$

but when  $n = \infty$ ,  $\frac{2\pi}{n} = 0$ , therefore the limit of  $\sin \frac{2\pi}{n} \div \frac{2\pi}{n}$  is unity.

Thus the required limit is  $\pi r^2$ .

$$13. \text{Lt.}_{\theta=0} \left( \frac{1 - \cos \theta}{\theta \sin \theta} \right) = \text{Lt.}_{\theta=0} \left( \frac{\tan \frac{\theta}{2}}{\theta} \right) = \text{Lt.}_{\theta=0} \left( \frac{1}{2} \cdot \frac{\tan \frac{\theta}{2}}{\frac{\theta}{2}} \right) = \frac{1}{2}.$$

$$14. \text{Lt.}_{\theta=0} \left( \frac{m \sin m\theta - n \sin n\theta}{\tan m\theta + \tan n\theta} \right) = \text{Lt.}_{\theta=0} \left( \frac{m \cdot m\theta - n \cdot n\theta}{m\theta + n\theta} \right)$$

$$= \frac{m^2 - n^2}{m + n} = m - n.$$

[Art. 268.]

$$15. \cos \left( \frac{\pi}{3} + \theta \right) = \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2} - \frac{\sqrt{3}}{200} = .491 \text{ nearly.}$$

$$16. 10' 30'' = \frac{21}{2 \times 60 \times 180} \times \frac{22}{7} = \frac{11}{3600} \text{ radians};$$

$$\therefore \sin 30^\circ 10' 30'' = \sin \left( \frac{\pi}{6} + \frac{11}{3600} \right)$$

$$= \frac{1}{2} \cos \frac{11}{3600} + \frac{\sqrt{3}}{2} \sin \frac{11}{3600}$$

$$= \frac{1}{2} + \frac{11\sqrt{3}}{7200} = .503 \text{ nearly.}$$

$$17. \cos\left(\frac{\pi}{3} + \theta\right) = .49; \text{ and } \cos\frac{\pi}{3} = .5.$$

$\therefore \theta$  is a very small angle, so that approximately

$$\cos\left(\frac{\pi}{3} + \theta\right) = \frac{1}{2} - \frac{\sqrt{3}}{2} \theta;$$

$$\therefore \frac{1}{2} - \frac{\sqrt{3}}{2} \theta = .49;$$

$$\therefore \frac{\sqrt{3}\theta}{2} = .01;$$

$$\therefore \theta = \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{150} \text{ radians}$$

$$= \frac{\sqrt{3}}{150} \times \frac{7}{22} \times 180^\circ = \frac{21\sqrt{3}}{55} \text{ degrees}$$

$$= 39.7' \text{ nearly.}$$

### EXAMPLES. XXI. b. PAGE 271.

1. Let the distance be  $x$  miles; then by the rule on page 269, we have

$$x^2 = \frac{3 \times 96}{2} = 3 \times 48;$$

$$\therefore x = 12;$$

that is, the distance is 12 miles.

2. Let  $a$  feet be the height of lighthouse above the sea level;

then  $15^2 = \frac{3a}{2}$ , or  $a = 150$ ;

that is, the height of lighthouse = 150 feet.

3. Let the distances in miles of the horizon visible from the masts of the ships be  $x_1, x_2$ ;

then  $x_1^2 = \frac{3 \times 32^2}{2} = 49; \therefore x_1 = 7,$

$$x_2^2 = \frac{3 \times 42^2}{2} = 64; \therefore x_2 = 8;$$

$\therefore$  the greatest distance at which one mast can be seen from the other

$$= x_1 + x_2 = 15 \text{ miles.}$$

4. Let the distances in miles of the horizon seen from the two masts be  $x, y$  respectively,

then 
$$x^2 = \frac{3 \times 54}{2} = 81; \therefore x = 9;$$

also 
$$x + y = 20; \text{ whence } y = 11.$$

Height of mast of second ship  $= \frac{2y^2}{3}$  feet  

$$= \frac{242}{3} \text{ feet} = 80 \text{ ft. } 8 \text{ in.}$$

5. Let  $x, y$  be the distances in miles of the horizon visible from the mast and the lamp respectively;

then 
$$x^2 = \frac{3 \times 73\frac{1}{2}}{2}; \therefore x = \frac{21}{2},$$

and 
$$x + y = 28; \text{ whence } y = \frac{35}{2}.$$

$\therefore$  height of lamp 
$$= \frac{2y^2}{3} = \frac{35^2}{6} = \frac{1225}{6} \text{ ft.}$$
  

$$= 204 \text{ ft. } 2 \text{ in.}$$

6. From the formula on page 271, we have

number of degrees in dip of horizon  $= \frac{10}{11} \sqrt{\frac{2 \times 2640}{3 \times 1760}} = \frac{10}{11};$

$\therefore$  dip of the horizon  $= 54' 33''$ , nearly.

7. The greatest distance at which the light must be visible is the distance of a point exactly opposite a point on the shore midway between two lighthouses, and  $3\frac{1}{2}$  miles from it.

This distance  $= \sqrt{12^2 + \left(\frac{7}{2}\right)^2} = \sqrt{\frac{625}{4}} = \frac{25}{2}$  miles.

$\therefore$  height of lamp  $= \frac{2}{3} \times \frac{625}{4} = \frac{625}{6}$  feet  $= 104 \text{ ft. } 2 \text{ in.}$

8. Let the height of the hill be  $h$  miles,

then we have 
$$1.81 = \frac{10}{11} \sqrt{2h};$$

$$\therefore \frac{20}{11} = \frac{10}{11} \sqrt{2h}; \text{ whence } h = 2;$$

that is, the height of the hill  $= 2 \text{ miles} = 10560 \text{ feet.}$

$$9. \text{ Height of hill} = \frac{2 \times (30 \cdot 25)^2}{3} \text{ feet} = 610 \text{ ft. nearly.}$$

$$\begin{aligned} \text{The dip of the horizon} &= \frac{10}{11} \sqrt{\frac{4 \times 121^2}{3 \times 16} \times \frac{1}{3 \times 1760}} \text{ degrees} \\ &= \frac{5}{12} \sqrt{\frac{11}{10}} \text{ degrees} \\ &= \frac{5}{2} \sqrt{110} \text{ minutes} \\ &= 26' 13'', \text{ nearly.} \end{aligned}$$

$$\begin{aligned} 10. \text{ We have } N &= \frac{10}{11} \sqrt{2h}, \text{ where } h = \text{height in miles,} \\ &= \frac{10}{11} \sqrt{\frac{2a}{3 \times 1760}}, \text{ where } a = \text{height in feet,} \\ &= \frac{10}{11} \sqrt{\frac{4r^2}{9 \times 1760}} \\ &= \frac{\pi}{66} \sqrt{\frac{10}{11}}. \end{aligned}$$

$$\begin{aligned} 11. \quad \frac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta} &= \frac{2 \sin 2\theta \cos 2\theta \cot \theta}{2 \sin^2 \theta \cot^2 2\theta} \\ &= \frac{\sin^3 2\theta \cos \theta}{\sin^3 \theta \cos 2\theta} = \frac{8 \cos^4 \theta}{\cos 2\theta}; \\ \therefore \text{Lt.}_{\theta=0} \left( \frac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta} \right) &= \text{Lt.}_{\theta=0} \left( \frac{8 \cos^4 \theta}{\cos 2\theta} \right) = 8. \end{aligned}$$

$$12. \quad \frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} = \frac{2 \sin^2 \frac{\theta}{2} + \sin \theta}{2 \sin^2 \frac{\theta}{2} - \sin \theta} = \frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}};$$

$$\therefore \text{Lt.}_{\theta=0} \left( \frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} \right) = \text{Lt.}_{\theta=0} \left( \frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}} \right) = -1.$$

$$13. (1) \quad Lt_{\theta=a} \left( \frac{\sin \theta - \sin a}{\theta - a} \right) = Lt_{\theta=a} \left( \frac{\sin \frac{\theta-a}{2} \cos \frac{\theta+a}{2}}{\frac{\theta-a}{2}} \right).$$

But 
$$Lt_{\theta=a} \left( \frac{\sin \frac{\theta-a}{2}}{\frac{\theta-a}{2}} \right) = 1,$$

[Art. 266.]

$$\therefore \text{required limit} = Lt_{\theta=a} \left( \cos \frac{\theta+a}{2} \right) = \cos a.$$

$$(2) \quad Lt_{\theta=a} \left( \frac{\cos \theta - \cos a}{\theta - a} \right) = Lt_{\theta=a} \left( \frac{\sin \frac{a-\theta}{2} \sin \frac{a+\theta}{2}}{\frac{\theta-a}{2}} \right)$$

$$= Lt_{\theta=a} \left( -\sin \frac{a+\theta}{2} \right) = -\sin a.$$

14. Let  $AB=32$ ,  $AC=31$ ;

then

$$\tan C = \frac{32}{31} = 1 + \frac{1}{31};$$

$\therefore C$  is a little greater than  $45^\circ$ ;

$$\therefore C = \frac{\pi}{4} + \theta, \text{ where } \theta \text{ is small;}$$

$$\therefore \frac{1+\theta}{1-\theta} = \frac{32}{31};$$

$$\therefore \theta = \frac{1}{63} \text{ radians} = \frac{1}{63} \times \frac{7 \times 180}{22} \text{ degrees} = \frac{10^\circ}{11} = 54' 33'';$$

$$\therefore C = 45^\circ 54' 33'', \quad B = 44^\circ 5' 27''.$$

15. We have

$$\angle BPA = a = \angle BAP;$$

$$\therefore AB = BP;$$

$$\therefore \frac{AB}{BC} = \frac{BP}{BC} = \frac{\sin 3a}{\sin a} = 3 - 4 \sin^2 a = 3,$$

since the object is distant and therefore  $a$  is small;

that is,

$$AB = 3BC, \text{ nearly.}$$

16. We shall first shew that  $\frac{\tan(\theta+h)}{\theta+h} - \frac{\tan \theta}{\theta}$  is positive,  $h$  being the radian measure of a small positive angle.

$$\text{This fraction} = \frac{\theta \tan(\theta+h) - (\theta+h) \tan \theta}{\theta(\theta+h)} = \frac{\theta(\tan \theta + h - \tan \theta) - h \tan \theta}{\theta(\theta+h)}$$

$$= \frac{\frac{\theta \sin h}{\cos \theta \cos(\theta+h)} - \frac{h \sin \theta}{\cos \theta}}{\theta(\theta+h)} = \frac{\theta \sin h - h \sin \theta \cos(\theta+h)}{\theta(\theta+h) \cos \theta \cos(\theta+h)}.$$

$$\text{Now} \quad \frac{\sin h}{h} > \frac{\sin \theta}{\theta} \text{ if } h < \theta \quad [\text{Art. 272.}]$$

$$\text{and} \quad \cos(\theta+h) < 1;$$

$$\therefore \theta \sin h > h \sin \theta \cos(\theta+h);$$

$$\text{that is, the fraction is positive, and } \therefore \frac{\tan(\theta+h)}{\theta+h} > \frac{\tan \theta}{\theta}.$$

$$\therefore \frac{\tan \theta}{\theta} \text{ continually increases as } \theta \text{ increases.}$$

$$\text{When } \theta=0, \frac{\tan \theta}{\theta}=1; \text{ when } \theta=\frac{\pi}{2}, \frac{\tan \theta}{\theta}=\infty.$$

Thus the proposition is established.

### MISCELLANEOUS EXAMPLES. H. PAGE 283.

$$1. \cos 2\alpha + \cos 2\beta + 2 \cos(\alpha + \beta) = 2 \cos(\alpha + \beta) \{ \cos(\alpha - \beta) + 1 \}$$

$$= 4 \cos(\alpha + \beta) \cos^2 \frac{\alpha - \beta}{2},$$

$$\sin 2\alpha + \sin 2\beta + 2 \sin(\alpha + \beta) = 2 \sin(\alpha + \beta) \{ \cos(\alpha - \beta) + 1 \}$$

$$= 4 \sin(\alpha + \beta) \cos^2 \frac{\alpha - \beta}{2};$$

$$\therefore \text{hypotenuse} = 4 \cos^2 \frac{\alpha - \beta}{2} \sqrt{\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta)} = 4 \cos^2 \frac{\alpha - \beta}{2}.$$

2. If the in-centre and circum-centre are at equal distances from  $BC$ ,

$$\text{we have} \quad R \cos A = r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2};$$

$$\therefore \cos A = \cos A + \cos B + \cos C - 1;$$

$$\therefore \cos B + \cos C = 1.$$

3. Let  $\theta$  be the required angle; then we have

$$\tan(45^\circ + \theta) = 2;$$

$$\therefore \log \tan(45^\circ + \theta) = .3010300$$

$$\begin{array}{r} \log \tan 63^\circ 26' = .3009994 \\ \text{diff.} \quad \quad \quad 306 \end{array}$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{306}{3159} \times 60'' = 5.8'' = 6'', \text{ nearly};$$

$$\therefore 45^\circ + \theta = 63^\circ 26' 6'';$$

$$\therefore \text{required angle } 18^\circ 26' 6''.$$

4. Denote each expression by  $E$ ; then

$$E^2 = (1 - \sin^2 \alpha)(1 - \sin^2 \beta)(1 - \sin^2 \gamma) = \cos^2 \alpha \cos^2 \beta \cos^2 \gamma;$$

that is,

$$E = \pm \cos \alpha \cos \beta \cos \gamma.$$

5. Let  $p_1, p_2$  be the distances of the chords from the centre, and let  $r$  be the radius; then

$$p_1 = r \cos 36^\circ;$$

similarly

$$p_2 = r \cos 72^\circ;$$

$$\therefore \text{distance between the chords} = p_1 - p_2 = r(\cos 36^\circ - \cos 72^\circ)$$

$$= r \left( \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} \right) = \frac{r}{2}.$$

$$\text{Also the sum of the squares of the chords} = 4r^2 \sin^2 36^\circ + 4r^2 \sin^2 72^\circ$$

$$= 4r^2 \left( 1 - \frac{3+\sqrt{5}}{8} + 1 - \frac{3-\sqrt{5}}{8} \right)$$

$$= 5r^2.$$

6. Let  $A$  be the point at which the railways meet, then we have to solve a triangle in which  $A = 60^\circ$ ,  $a = 43$ ,  $b = 48$ . It is easy to see that the solution is ambiguous; hence from the third figure of Art. 148 we have

$$CD = 48 \sin 60^\circ = 24\sqrt{3}, \quad AD = 48 \cos 60^\circ = 24.$$

Also

$$B_2D = \sqrt{43^2 - (24\sqrt{3})^2} = \sqrt{121} = 11.$$

$$\therefore AB_1 = 24 + 11 = 35 \text{ miles,}$$

$$AB_2 = 24 - 11 = 13 \text{ miles.}$$

7.

$$\alpha = \cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b};$$

$$\therefore \cos \alpha = \frac{xy}{ab} - \sqrt{\left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{y^2}{b^2}\right)};$$

$$\therefore \cos^2 \alpha - \frac{2xy}{ab} \cos \alpha + \frac{x^2 y^2}{a^2 b^2} = \left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{y^2}{b^2}\right);$$

$$\therefore 1 - \cos^2 \alpha = \frac{x^2}{a^2} - \frac{2xy}{ab} \cos \alpha + \frac{y^2}{b^2};$$

that is,

$$\sin^2 \alpha = \frac{x^2}{a^2} - \frac{2xy}{ab} \cos \alpha + \frac{y^2}{b^2}.$$

8. We have  $p = 4R \cos \frac{A}{2}$ ,  $q = 4R \cos \frac{B}{2}$ ,  $r = 4R \cos \frac{C}{2}$ ;

$$\therefore \frac{a}{p} = \frac{2R \sin A}{4R \cos \frac{A}{2}} = \sin \frac{A}{2};$$

$$\therefore \frac{a^2}{p^2} + \frac{b^2}{q^2} + \frac{c^2}{r^2} + \frac{2abc}{pqr} = \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1.$$

[Ex. 13, p. 120.]

9. Let  $P$  be the top of the tower, and let  $x$  be its height;

then

$$PA = \frac{x}{\sin \alpha}, \quad PB = \frac{x}{\sin \beta};$$

and

$$PA^2 + OA^2 = PO^2 = PB^2 + OB^2;$$

$$\therefore \frac{x^2}{\sin^2 \alpha} + a^2 = \frac{x^2}{\sin^2 \beta} + b^2;$$

$$\therefore x^2 = \frac{(a^2 - b^2) \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha - \sin^2 \beta} = \frac{(a^2 - b^2) \sin^2 \alpha \sin^2 \beta}{\sin(\alpha + \beta) \sin(\alpha - \beta)};$$

hence the height of the tower

$$= \frac{\sqrt{a^2 - b^2} \sin \alpha \sin \beta}{\sqrt{\sin(\alpha + \beta) \sin(\alpha - \beta)}}.$$

10. This example follows readily from the results proved in Examples 18, 24, 25 of XVIII. a.

$$\begin{aligned} r^2 + r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3 - r)^2 + 2r(r_1 + r_2 + r_3) - 2(r_2 r_3 + r_3 r_1 + r_1 r_2) \\ &= 16R^2 + 2\{ (r_1 r_2 + r r_3) + (r_2 r_3 + r r_1) + (r_3 r_1 + r r_2) \} \\ &\quad - 4(r_1 r_2 + r_2 r_3 + r_3 r_1) \\ &= 16R^2 + 2(ab + bc + ca) - 4s^2 \\ &= 16R^2 - a^2 - b^2 - c^2. \end{aligned}$$

11. (1) Let  $\angle BAD = \theta$ ; then  $\angle CAD = A - \theta$ ;

$$\therefore \frac{\sin(A - \theta)}{\sin(A + B)} = \frac{CD}{AD} = \frac{BD}{AD} = \frac{\sin \theta}{\sin B};$$

$$\therefore \frac{\sin(A - \theta)}{\sin \theta} = \frac{\sin(A + B)}{\sin B};$$

$$\therefore \cot \theta - \cot A = \cot A + \cot B;$$

$$\cot BAD = 2 \cot A + \cot B.$$

that is,

(2) Draw  $AM$  perpendicular to  $BC$ , then

$$\begin{aligned} 2 \cot ADC &= \frac{2DM}{AM} = \frac{2DC - 2MC}{AM} \\ &= \frac{(BC - MC) - MC}{AM} = \frac{BM}{AM} - \frac{MC}{AM} \\ &= \cot B - \cot C. \end{aligned}$$

12. See fig. of Art. 223.

Then  $\frac{a}{p} = \frac{BG + GC}{OG} = \frac{BG}{OG} + \frac{GC}{OG} = \tan C + \tan B;$

$$\therefore \frac{a}{p} + \frac{b}{q} - \frac{c}{r} = 2 \tan C,$$

and  $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 2 (\tan A + \tan B + \tan C);$

$$\begin{aligned} \therefore 4 \left( \frac{a}{p} + \frac{b}{q} + \frac{c}{r} \right) &= 8 \tan A \tan B \tan C \\ &= \left( \frac{a}{p} + \frac{b}{q} - \frac{c}{r} \right) \left( \frac{b}{q} + \frac{c}{r} - \frac{a}{p} \right) \left( \frac{c}{r} + \frac{a}{p} - \frac{b}{q} \right) \end{aligned}$$

### EXAMPLES. XXIII. a. PAGE 291.

1. Here the common difference is  $2a$ ;

$$\therefore S = \frac{\sin na}{\sin a} \sin \frac{a + 2n - 1 a}{2} = \frac{\sin^2 na}{\sin a}.$$

2. Here the common difference is  $-\beta$ ;

$$\therefore S = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \frac{a + a - n - 1 \beta}{2} = \frac{\sin \frac{n\beta}{2} \cos \left( a - \frac{n-1}{2} \beta \right)}{\sin \frac{\beta}{2}}.$$

3. Here the common difference is  $-\frac{\pi}{n}$ ;

$$\therefore S = \frac{\sin \frac{\pi}{2}}{\sin \frac{\pi}{2n}} \sin \frac{a + a - (n-1)\frac{\pi}{n}}{2} = \frac{-\cos \left(a + \frac{\pi}{2n}\right)}{\sin \frac{\pi}{2n}}.$$

4. Here the common difference is  $\frac{\pi}{k}$ ;

$$\therefore S = \frac{\sin \frac{n\pi}{2k}}{\sin \frac{\pi}{2k}} \cos \frac{\frac{\pi}{k} + \frac{n\pi}{k}}{2} = \frac{\sin \frac{n\pi}{2k} \cos \frac{(n+1)\pi}{2k}}{\sin \frac{\pi}{2k}}.$$

5. The common difference is  $\frac{2\pi}{19}$  and the number of terms is 9;

$$\therefore S = \frac{\sin \frac{9\pi}{19}}{\sin \frac{\pi}{19}} \cos \frac{9\pi}{19} = \frac{\sin \frac{18\pi}{19}}{2 \sin \frac{\pi}{19}} = \frac{\sin \left(\pi - \frac{\pi}{19}\right)}{2 \sin \frac{\pi}{19}} = \frac{1}{2}.$$

6. Here the common difference is  $\frac{2\pi}{21}$  and the number of terms is 10;

$$\therefore S = \frac{\sin \frac{10\pi}{21}}{\sin \frac{\pi}{21}} \cos \frac{11\pi}{21} = \frac{\sin \frac{22\pi}{21}}{2 \sin \frac{\pi}{21}} = -\frac{1}{2}.$$

7. The common difference is  $\frac{\pi}{n}$ ;

$$\begin{aligned} \therefore S &= \frac{\sin \frac{(n-1)\pi}{2n}}{\sin \frac{\pi}{2n}} \sin \frac{\pi + (n-1)\pi}{2n} \\ &= \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{2n}\right)}{\sin \frac{\pi}{2n}} \sin \frac{\pi}{2} = \cot \frac{\pi}{2n}. \end{aligned}$$

8. The common difference is  $\frac{2\pi}{n}$ ;

$$\begin{aligned}\therefore S &= \frac{\sin \frac{(2n-1)\pi}{n}}{\sin \frac{\pi}{n}} \cos \frac{\pi + \{2(2n-1) - 1\}\pi}{2n} \\ &= \frac{\sin \left(2\pi - \frac{\pi}{n}\right)}{\sin \frac{\pi}{n}} \cos \frac{(4n-2)\pi}{2n} = -\cos \frac{\pi}{n}.\end{aligned}$$

9. The common difference is  $-a$ ;

$$\therefore S = \frac{\sin na}{\sin \frac{a}{2}} \sin \frac{na + (n-2n-1)a}{2} = \sin na.$$

10. The series may be written

$$\sin \theta + \sin (\pi + 2\theta) + \sin (2\pi + 3\theta) + \sin (3\pi + 4\theta) + \dots$$

$$\begin{aligned}\therefore S &= \frac{\sin \frac{n(\pi + \theta)}{2}}{\sin \frac{\pi + \theta}{2}} \sin \frac{\theta + n-1\pi + n\ell}{2} \\ &= \frac{\sin \frac{n(\pi + \theta)}{2} \sin \left\{ \frac{(n+1)\theta}{2} + \frac{(n-1)\pi}{2} \right\}}{\sin \frac{\pi + \theta}{2}}.\end{aligned}$$

11. The series may be written

$$\cos \alpha + \cos (\pi + \alpha - \beta) + \cos (2\pi + \alpha - 2\beta) + \dots$$

$$\begin{aligned}\therefore S &= \frac{\sin \frac{n(\pi - \beta)}{2}}{\sin \frac{\pi - \beta}{2}} \cos \frac{\alpha + (n-1)(\pi - \beta) + \alpha}{2} \\ &= \frac{\sin \frac{n(\pi - \beta)}{2} \cos \left\{ \alpha + \frac{(n-1)(\pi - \beta)}{2} \right\}}{\sin \frac{\pi - \beta}{2}}.\end{aligned}$$

12. The series may be written

$$\cos \alpha + \cos \left( \frac{\pi}{2} + \alpha - \beta \right) + \cos (\pi + \alpha - 2\beta) + \cos \left( \frac{3\pi}{2} + \alpha - 3\beta \right) + \dots$$

$$\begin{aligned} \therefore S &= \frac{\sin \frac{n(\pi - 2\beta)}{4}}{\sin \frac{\pi - 2\beta}{4}} \cos \frac{2\alpha + (n-1) \left( \frac{\pi}{2} - \beta \right)}{2} \\ &= \frac{\sin \frac{n(\pi - 2\beta)}{4} \cos \left\{ \alpha + \frac{(n-1)(\pi - 2\beta)}{4} \right\}}{\sin \frac{\pi - 2\beta}{4}}. \end{aligned}$$

$$13. \quad S = \frac{1}{2} \{ (\cos \theta - \cos 3\theta) + (\cos \theta - \cos 5\theta) + (\cos \theta - \cos 7\theta) + \dots \}$$

$$= \frac{1}{2} \{ n \cos \theta - (\cos 3\theta + \cos 5\theta + \cos 7\theta + \dots + \cos \overline{2n+1}\theta) \}$$

$$= \frac{n \cos \theta}{2} - \frac{\sin n\theta}{2 \sin \theta} \cos (n+2)\theta.$$

$$14. \quad S = \frac{1}{2} \{ (\sin 4\alpha - \sin 2\alpha) + (\sin 8\alpha - \sin 2\alpha) + (\sin 12\alpha - \sin 2\alpha) + \dots \}$$

$$= \frac{1}{2} \{ (\sin 4\alpha + \sin 8\alpha + \sin 12\alpha + \dots + \sin 4n\alpha) - n \sin 2\alpha \}$$

$$= \frac{\sin 2n\alpha}{2 \sin 2\alpha} \sin 2(n+1)\alpha - \frac{n \sin 2\alpha}{2}.$$

$$15. \quad \sec \alpha \sec 2\alpha = \frac{1}{\cos \alpha \cos 2\alpha} = \operatorname{cosec} \alpha \cdot \frac{\sin (2\alpha - \alpha)}{\cos \alpha \cos 2\alpha}$$

$$= \operatorname{cosec} \alpha (\tan 2\alpha - \tan \alpha).$$

Similarly,

$$\sec 2\alpha \sec 3\alpha = \operatorname{cosec} \alpha (\tan 3\alpha - \tan 2\alpha),$$

.....

$$\sec n\alpha \sec (n+1)\alpha = \operatorname{cosec} \alpha \{ \tan (n+1)\alpha - \tan n\alpha \}.$$

By addition,

$$S = \operatorname{cosec} \alpha \{ \tan (n+1)\alpha - \tan n\alpha \}.$$

$$16. \quad \begin{aligned} \operatorname{cosec} \theta \operatorname{cosec} 3\theta &= \operatorname{cosec} 2\theta (\cot \theta - \cot 3\theta), \\ \operatorname{cosec} 3\theta \operatorname{cosec} 5\theta &= \operatorname{cosec} 2\theta (\cot 3\theta - \cot 5\theta), \\ &\dots\dots\dots \end{aligned}$$

$$\operatorname{cosec} (2n-1)\theta \operatorname{cosec} (2n+1)\theta = \operatorname{cosec} 2\theta \{\cot (2n-1)\theta - \cot (2n+1)\theta\}.$$

$$\text{By addition,} \quad S = \operatorname{cosec} 2\theta \{\cot \theta - \cot (2n+1)\theta\}.$$

$$17. \quad \tan \frac{a}{2} \sec a = \tan a - \tan \frac{a}{2},$$

$$\tan \frac{a}{2^2} \sec \frac{a}{2} = \tan \frac{a}{2} - \tan \frac{a}{2^2},$$

$$\dots\dots\dots \tan \frac{a}{2^n} \sec \frac{a}{2^{n-1}} = \tan \frac{a}{2^{n-1}} - \tan \frac{a}{2^n}.$$

$$\text{By addition,} \quad S = \tan a - \tan \frac{a}{2^n}.$$

$$18. \quad \cos 2a \operatorname{cosec} 3a = \frac{\cos 2a \sin a}{\sin 3a \sin a} = \frac{1}{2} \cdot \frac{\sin 3a - \sin a}{\sin 3a \sin a}$$

$$= \frac{1}{2} (\operatorname{cosec} a - \operatorname{cosec} 3a),$$

$$\cos 6a \operatorname{cosec} 9a = \frac{1}{2} (\operatorname{cosec} 3a - \operatorname{cosec} 9a),$$

$$\dots\dots\dots \cos 3^{n-1} \cdot 2a \operatorname{cosec} 3^n a = \frac{1}{2} (\operatorname{cosec} 3^{n-1} a - \operatorname{cosec} 3^n a).$$

$$\text{By addition,} \quad S = \frac{1}{2} \operatorname{cosec} a - \operatorname{cosec} 3^n a.$$

$$19. \quad \sin a \sec 3a = \frac{\sin a}{\cos 3a} = \frac{2 \sin a \cos a}{2 \cos 3a \cos a}$$

$$= \frac{\sin (3a - a)}{2 \cos 3a \cos a}$$

$$= \frac{1}{2} (\tan 3a - \tan a),$$

$$\sin 3a \sec 9a = \frac{1}{2} (\tan 9a - \tan 3a),$$

$$\dots\dots\dots \sin 3^{n-1} a \sec 3^n a = \frac{1}{2} (\tan 3^n a - \tan 3^{n-1} a).$$

$$\text{By addition,} \quad S = \frac{1}{2} (\tan 3^n a - \tan a).$$

20. Let  $AB$  be the diameter,  $P_1, P_2, P_3 \dots P_{n-1}$  the points of division of the arc of the semicircle starting from the end  $B$ ; then

$$AP_1 = 2a \cos \frac{\pi}{2n}, \quad AP_2 = 2a \cos \frac{2\pi}{2n}, \dots, AP_{n-1} = 2a \cos \frac{(n-1)\pi}{2n};$$

$$\begin{aligned} \therefore \text{sum of distances} &= 2a \left\{ \cos \frac{\pi}{2n} + \cos \frac{2\pi}{2n} + \cos \frac{3\pi}{2n} + \dots + \cos \frac{(n-1)\pi}{2n} \right\} \\ &= \frac{2a \sin \frac{(n-1)\pi}{4n}}{\sin \frac{\pi}{4n}} \cos \frac{\pi}{4} \\ &= \frac{a \left\{ \sin \left( \frac{\pi}{2} - \frac{\pi}{4n} \right) - \sin \frac{\pi}{4n} \right\}}{\sin \frac{\pi}{4n}} \\ &= a \left( \cot \frac{\pi}{4n} - 1 \right). \end{aligned}$$

21. Let  $P_1, P_2, P_3 \dots$  be the angular points of the polygon beginning with that nearest to  $XX'$  in the quadrant  $XOY$ .

Let  $p_1, p_2, p_3 \dots$  be perpendiculars from  $P_1, P_2, P_3 \dots$  on  $XX'$ , and  $q_1, q_2, q_3 \dots$  be perpendiculars from  $P_1, P_2, P_3 \dots$  on  $YY'$ .

Let  $\angle P_1OX = \theta$ , and let  $r$  be the radius of the circle;

then  $p_1 = r \sin \theta, p_2 = r \sin \left( \theta + \frac{2\pi}{n} \right), p_3 = r \sin \left( \theta + \frac{4\pi}{n} \right), \dots;$

$$\begin{aligned} \therefore S_p &= r \left\{ \sin \theta + \sin \left( \theta + \frac{2\pi}{n} \right) + \sin \left( \theta + \frac{4\pi}{n} \right) + \dots \text{to } n \text{ terms} \right\} \\ &= r \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \sin \frac{2\theta + (n-1)\frac{2\pi}{n}}{2} = 0. \end{aligned}$$

Similarly  $S_q = 0$ .

### EXAMPLES. XXIII. b. PAGE 294.

1.  $2S = 1 + \cos 2\theta + 1 + \cos 6\theta + 1 + \cos 10\theta + \dots$

$$= n + \frac{\sin 2n\theta}{\sin 2\theta} \cos \frac{2\theta + 2\theta + (n-1)4\theta}{2};$$

$$\therefore S = \frac{n}{2} + \frac{\sin 4n\theta}{4 \sin 2\theta}.$$

$$2. \quad 2S = 1 - \cos 2\alpha + 1 - \cos 2\left(\alpha + \frac{\pi}{n}\right) + 1 - \cos 2\left(\alpha + \frac{2\pi}{n}\right) + \dots$$

$$= n - \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{4\alpha + (n-1)\frac{2\pi}{n}}{2} = n.$$

$$3. \quad 2S = 1 + \cos 2\alpha + 1 + \cos 2\left(\alpha - \frac{\pi}{n}\right) + 1 + \cos 2\left(\alpha - \frac{2\pi}{n}\right) + \dots$$

$$= n + \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{4\alpha - (n-1)\frac{2\pi}{n}}{2} = n.$$

$$4. \quad 4S = 3 \sin \theta - \sin 3\theta + 3 \sin 2\theta - \sin 6\theta + 3 \sin 3\theta - \sin 9\theta + \dots$$

$$= 3 \{ \sin \theta + \sin 2\theta + \sin 3\theta + \dots \} - (\sin 3\theta + \sin 6\theta + \sin 9\theta + \dots)$$

$$= \frac{3 \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \sin \frac{(n+1)\theta}{2} - \frac{\sin \frac{3n\theta}{2}}{\sin \frac{3\theta}{2}} \sin \frac{3(n+1)\theta}{2};$$

$$\therefore S = \frac{3 \sin \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{4 \sin \frac{\theta}{2}} - \frac{\sin \frac{3n\theta}{2} \sin \frac{3(n+1)\theta}{2}}{\sin \frac{3\theta}{2}}.$$

$$5. \quad 4S = 3 \left\{ \sin \alpha + \sin \left( \alpha + \frac{2\pi}{n} \right) + \sin \left( \alpha + \frac{4\pi}{n} \right) + \dots \right\}$$

$$- \left\{ \sin 3\alpha + \sin 3 \left( \alpha + \frac{2\pi}{n} \right) + \sin 3 \left( \alpha + \frac{4\pi}{n} \right) + \dots \right\}$$

$$= 3 \frac{\sin \pi}{\sin \frac{\pi}{n}} \sin \frac{2\alpha + (n-1)\frac{2\pi}{n}}{2} - \frac{\sin 3\pi}{\sin \frac{3\pi}{n}} \sin \frac{6\alpha + (n-1)\frac{6\pi}{n}}{2}$$

$$= 0.$$

$$\begin{aligned}
 6. \quad 4S &= 3 \left\{ \cos a + \cos \left( a - \frac{2\pi}{n} \right) + \cos \left( a - \frac{4\pi}{n} \right) + \dots \right\} \\
 &\quad + \cos 3a + \cos 3 \left( a - \frac{2\pi}{n} \right) + \cos 3 \left( a - \frac{4\pi}{n} \right) + \dots \\
 &= 3 \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{2a - (n-1) \frac{2\pi}{n}}{2} + \frac{\sin 3\pi}{\sin \frac{3\pi}{n}} \cos \frac{6a - (n-1) \frac{6\pi}{n}}{2} \\
 &= 0.
 \end{aligned}$$

$$7. \quad \text{We have} \quad \tan \theta = \cot \theta - 2 \cot 2\theta, \quad [\text{XI. d. Ex. 18.}]$$

$$2 \tan 2\theta = 2 \cot 2\theta - 2^2 \cot 2^2 \theta,$$

$$2^2 \tan 2^2 \theta = 2^2 \cot 2^2 \theta - 2^3 \cot 2^3 \theta,$$

$$\dots\dots\dots$$

$$2^{n-1} \tan 2^{n-1} \theta = 2^{n-1} \cot 2^{n-1} \theta - 2^n \cot 2^n \theta;$$

$$\therefore \text{by addition,} \quad S = \cot \theta - 2^n \cot 2^n \theta.$$

$$\begin{aligned}
 8. \quad S &= \frac{1}{2 \cos a \cos 2a} + \frac{1}{2 \cos 2a \cos 3a} + \frac{1}{2 \cos 3a \cos 4a} + \dots \\
 &= \frac{1}{2} \{ \sec a \sec 2a + \sec 2a \sec 3a + \sec 3a \sec 4a + \dots \} \\
 &= \frac{1}{2} \operatorname{cosec} a \{ \tan (n+1) a - \tan a \}. \quad [\text{XXIII. a. Ex. 15.}]
 \end{aligned}$$

$$9. \quad \sin^2 \theta \sin 2\theta = \frac{\sin 2\theta}{2} (1 - \cos 2\theta);$$

$$\therefore \sin^2 \theta \sin 2\theta = \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4}.$$

Replacing  $\theta$  by  $2\theta$ , we obtain

$$\frac{1}{2} \sin^2 2\theta \sin 4\theta = \frac{\sin 4\theta}{4} - \frac{\sin 8\theta}{8}.$$

$$\text{Similarly,} \quad \frac{1}{4} \sin^2 4\theta \sin 8\theta = \frac{\sin 8\theta}{8} - \frac{\sin 16\theta}{16};$$

$$\dots\dots\dots$$

$$\frac{1}{2^{n-1}} \sin^2 2^{n-1} \theta \sin 2^n \theta = \frac{\sin 2^n \theta}{2^n} - \frac{\sin 2^{n+1} \theta}{2^{n+1}};$$

$$\therefore \text{by addition,} \quad S = \frac{\sin 2\theta}{2} - \frac{\sin 2^{n+1} \theta}{2^{n+1}}.$$

$$10. \quad 2 \cos \theta \sin^2 \frac{\theta}{2} = \cos \theta (1 - \cos \theta) = 1 - \cos^2 \theta = (1 - \cos \theta);$$

$$\therefore 2 \cos \theta \sin^2 \frac{\theta}{2} = \sin^2 \theta - 2 \sin^2 \frac{\theta}{2}.$$

Replacing  $\theta$  by  $\frac{\theta}{2}$ , we obtain

$$2^2 \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2^2} = 2 \sin^2 \frac{\theta}{2} - 2^2 \sin^2 \frac{\theta}{2^2}.$$

$$\text{Similarly,} \quad 2^3 \cos \frac{\theta}{2^2} \sin^2 \frac{\theta}{2^3} = 2^2 \sin^2 \frac{\theta}{2^2} - 2^3 \sin^2 \frac{\theta}{2^3};$$

$$\dots\dots\dots$$

$$2^n \cos \frac{\theta}{2^{n-1}} \sin^2 \frac{\theta}{2^n} = 2^{n-1} \sin^2 \frac{\theta}{2^{n-1}} - 2^n \sin^2 \frac{\theta}{2^n};$$

$$S = \sin^2 \theta - 2^n \sin^2 \frac{\theta}{2^n}.$$

$\therefore$  by addition,

11. We have

$$\tan^{-1} \frac{x}{n(n+1)+x^2} = \tan^{-1} \frac{\frac{x}{n} - \frac{x}{n+1}}{1 + \frac{x^2}{n(n+1)}} = \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1},$$

and hence

$$\tan^{-1} \frac{x}{1 \cdot 2 + x^2} = \tan^{-1} x - \tan^{-1} \frac{x}{2};$$

$$\tan^{-1} \frac{x}{2 \cdot 3 + x^2} = \tan^{-1} \frac{x}{2} - \tan^{-1} \frac{x}{3};$$

$$\dots\dots\dots$$

$$\tan^{-1} \frac{x}{n(n+1)+x^2} = \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1};$$

$$S = \tan^{-1} x - \tan^{-1} \frac{x}{n+1}.$$

$\therefore$  by addition,

$$12. \quad \tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} \frac{(n+1)-n}{1+n(n+1)} \\ = \tan^{-1} (n+1) - \tan^{-1} n.$$

$$\therefore \tan^{-1} \frac{1}{1+1+1^2} = \tan^{-1} 2 - \tan^{-1} 1;$$

$$\tan^{-1} \frac{1}{1+2+2^2} = \tan^{-1} 3 - \tan^{-1} 2;$$

$$\dots\dots\dots$$

$$\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} (n+1) - \tan^{-1} n;$$

$$\therefore S = \tan^{-1} (n+1) - \tan^{-1} 1 = \tan^{-1} (n+1) - \frac{\pi}{4}.$$

$$13. \quad \tan^{-1} \frac{2n}{2+n^2+n^4} = \tan^{-1} \frac{(1+n+n^2) - (1-n+n^2)}{1 + (1+n+n^2)(1-n+n^2)} \\ = \tan^{-1} (1+n+n^2) - \tan^{-1} (1-n+n^2).$$

$$\therefore \tan^{-1} \frac{2}{2+1^2+1^4} = \tan^{-1} 3 - \tan^{-1} 1;$$

$$\tan^{-1} \frac{4}{2+2^2+2^4} = \tan^{-1} 6 - \tan^{-1} 3;$$

.....

$$\tan^{-1} \frac{2n}{2+n^2+n^4} = \tan^{-1} (1+n+n^2) - \tan^{-1} (1-n+n^2);$$

$$\therefore S = \tan^{-1} (1+n+n^2) - \tan^{-1} 1$$

$$= \tan^{-1} (1+n+n^2) - \frac{\pi}{4}.$$

$$14. \quad \tan^{-1} \frac{2n}{1-n^2+n^4} = \tan^{-1} \frac{n^2+n - (n^2-n)}{1 + (n^2+n)(n^2-n)} \\ = \tan^{-1} (n^2+n) - \tan^{-1} (n^2-n).$$

$$\therefore \tan^{-1} \frac{2}{1-1^2+1^4} = \tan^{-1} 2 - \tan^{-1} 0;$$

$$\tan^{-1} \frac{4}{1-2^2+2^4} = \tan^{-1} 6 - \tan^{-1} 2;$$

.....

$$\tan^{-1} \frac{2n}{1-n^2+n^4} = \tan^{-1} (n^2+n) - \tan^{-1} (n^2-n);$$

$$\therefore S = \tan^{-1} (n^2+n).$$

15. Let  $O$  be the point on the circumference of the circle, and  $P, Q, R, \dots$  the vertices of the polygon beginning with that nearest to  $O$ . Let  $L$  be the other extremity of the diameter through  $O$ , and let  $\angle OLP = \theta$ ;

then  $OP = 2r \sin \theta$ ,  $OQ = 2r \sin \left( \theta + \frac{\pi}{n} \right)$ ,  $OR = 2r \sin \left( \theta + \frac{2\pi}{n} \right)$ , ..

$\therefore$  sum of the squares of the chords

$$= 4r^2 \left\{ \sin^2 \theta + \sin^2 \left( \theta + \frac{\pi}{n} \right) + \sin^2 \left( \theta + \frac{2\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= 2r^2 \left\{ n - \cos 2\theta - \cos 2 \left( \theta + \frac{\pi}{n} \right) - \cos 2 \left( \theta + \frac{2\pi}{n} \right) - \dots \right\}$$

$$= 2nr^2 - 2r^2 \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cos \left( 2\theta + (n-1) \frac{\pi}{n} \right)$$

$$= 2nr^2.$$

16. Let  $O$  be the centre of the inscribed circle, and let  $\angle PON_1 = \theta$ , where  $ON_1$  is the perpendicular from  $O$  parallel to  $PA_1$ ; let  $OP = x$ ;

$$\text{then } PA_1 = r - x \cos \theta, \quad PA_2 = r - x \cos \left( \theta + \frac{\pi}{n} \right),$$

$$PA_3 = r - x \cos \left( \theta + \frac{2\pi}{n} \right), \quad PA_4 = r - x \cos \left( \theta + \frac{3\pi}{n} \right), \dots$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n-1}$$

$$= nr - x \left\{ \cos \theta + \cos \left( \theta + \frac{2\pi}{n} \right) + \cos \left( \theta + \frac{4\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\} \\ = nr. \quad [\text{Art. 297.}]$$

$$\text{Similarly } PA_2 + PA_4 + \dots + PA_{2n}$$

$$= nr - x \left\{ \cos \left( \theta + \frac{\pi}{n} \right) + \cos \left( \theta + \frac{3\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\} \\ = nr.$$

17. Let  $Q$  be the other extremity of the diameter through  $P$ , and let  $\angle PQA_1 = \theta$ , and let  $r$  be the radius of the circle; then

$$PA_1 = 2r \sin \theta, \quad PA_2 = 2r \sin \left( \theta + \frac{\pi}{2n+1} \right), \quad PA_3 = 2r \sin \left( \theta + \frac{2\pi}{2n+1} \right), \dots$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n+1}$$

$$= 2r \left\{ \sin \theta + \sin \left( \theta + \frac{2\pi}{2n+1} \right) + \sin \left( \theta + \frac{4\pi}{2n+1} \right) + \dots \text{ to } n+1 \text{ terms} \right\} \\ = 2r \frac{\sin \frac{(n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \left( \theta + \frac{n\pi}{2n+1} \right);$$

$$\text{and } PA_2 + PA_4 + \dots + PA_{2n}$$

$$= 2r \left\{ \sin \left( \theta + \frac{\pi}{2n+1} \right) + \sin \left( \theta + \frac{3\pi}{2n+1} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= 2r \frac{\sin \frac{n\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \frac{1}{2} \left( 2\theta + \frac{\pi}{2n+1} + \frac{(2n-1)\pi}{2n+1} \right)$$

$$= 2r \frac{\sin \frac{(n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \left( \theta + \frac{n\pi}{2n+1} \right).$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n+1} = PA_2 + PA_4 + \dots + PA_{2n}.$$

18. Let  $p_1, p_2, p_3, \dots, p_n$  be the perpendiculars; then as in Ex. 16, we have

$$p_1 = r - r \cos \theta, \quad p_2 = r - r \cos \left( \theta + \frac{2\pi}{n} \right), \quad p_3 = r - r \cos \left( \theta + \frac{4\pi}{n} \right), \dots$$

$$\begin{aligned} \text{(i)} \quad \Sigma p^2 &= \{r - r \cos \theta\}^2 + \left\{r - r \cos \left( \theta + \frac{2\pi}{n} \right)\right\}^2 + \dots \text{ to } n \text{ terms} \\ &= nr^2 - 2r^2 \left\{ \cos \theta + \cos \left( \theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &\quad + r^2 \left\{ \cos^2 \theta + \cos^2 \left( \theta + \frac{2\pi}{n} \right) + \dots \right\}. \end{aligned}$$

$$\text{Now} \quad \cos \theta + \cos \left( \theta + \frac{2\pi}{n} \right) + \dots = 0; \quad [\text{Art. 297.}]$$

$$\begin{aligned} \therefore \Sigma p^2 &= nr^2 + \frac{r^2}{2} \left\{ n + \cos 2\theta + \cos 2 \left( \theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &= nr^2 + \frac{nr^2}{2} = \frac{3nr^2}{2}. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \Sigma p^3 &= nr^3 - 3r^3 \left\{ \cos \theta + \cos \left( \theta + \frac{2\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\} \\ &\quad + \frac{3r^3}{2} \left\{ n + \cos 2\theta + \cos 2 \left( \theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &\quad - \frac{r^3}{4} \left\{ 3 \cos \theta + 3 \cos \left( \theta + \frac{2\pi}{n} \right) + \dots \right. \\ &\quad \left. + \cos 3\theta + \cos 3 \left( \theta + \frac{3\pi}{n} \right) + \dots \right\} \\ &= nr^3 + \frac{3nr^3}{2} = \frac{5nr^3}{2}, \end{aligned}$$

since all the series of cosines vanish by Art. 297.

### EXAMPLES. XXIV. a. PAGE 301.

1. We have

$$\frac{1}{a} \cos \alpha + \frac{1}{b} \sin \alpha = \frac{1}{c}, \quad \frac{1}{a} \cos \beta + \frac{1}{b} \sin \beta = \frac{1}{c};$$

and the required result follows at once by cross multiplication as in Ex. 1, p. 297.

$$\begin{aligned} 4. \quad \cos (\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2}, \end{aligned}$$

as in Example 2, page 297.

$$\begin{aligned}
 5. \quad \cos^2 \frac{\alpha - \beta}{2} &= \frac{1 + \cos(\alpha - \beta)}{2} = \frac{1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta}{2} \\
 &= \frac{(a^2 + b^2) + (c^2 - b^2) + (c^2 - a^2)}{2(a^2 + b^2)} \\
 &= \frac{c^2}{a^2 + b^2}.
 \end{aligned}$$

$$6. \quad \sin 2\alpha + \sin 2\beta = 2 \sin(\alpha + \beta) \cos(\alpha - \beta).$$

From Example 4, we find that  $\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$ ;

$$\text{and} \quad \cos(\alpha - \beta) = \frac{c^2 - b^2}{a^2 + b^2} + \frac{c^2 - a^2}{a^2 + b^2} = \frac{2c^2 - a^2 - b^2}{a^2 + b^2}.$$

$$\begin{aligned}
 7. \quad \sin^2 \alpha + \sin^2 \beta &= (\sin \alpha + \sin \beta)^2 - 2 \sin \alpha \sin \beta \\
 &= \left( \frac{2bc}{a^2 + b^2} \right)^2 - \frac{2(c^2 - a^2)}{a^2 + b^2}. \quad [\text{See p. 298.}]
 \end{aligned}$$

8. Here  $\alpha$  and  $\beta$  are solutions of  $a \cos \theta + b \sin \theta = c$ ; hence as in Example 4, we find

$$\cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}, \text{ and therefore } \sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}.$$

$$\begin{aligned}
 \text{Again,} \quad \cot \alpha + \cot \beta &= \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} \\
 &= \frac{2ab}{a^2 + b^2} \div \frac{c^2 - a^2}{a^2 + b^2}. \quad [\text{See p. 298.}]
 \end{aligned}$$

9. By squaring and adding, we have

$$2 + 2 \cos(\theta - \phi) = a^2 + b^2.$$

$$\text{Again,} \quad \cos(\theta + \phi) = \frac{a^2 - b^2}{a^2 + b^2}. \quad [\text{Ex. 4, p. 299.}]$$

And

$$\begin{aligned}
 2 \cos \theta \cos \phi &= \cos(\theta + \phi) + \cos(\theta - \phi); \\
 \therefore 4 \cos \theta \cos \phi &= \frac{2(a^2 - b^2)}{a^2 + b^2} + a^2 + b^2 - 2 \\
 &= \frac{(a^2 + b^2)^2 - 4b^2}{a^2 + b^2}.
 \end{aligned}$$

10.

$$\begin{aligned}
 \cos 2\theta + \cos 2\phi &= 2 \cos(\theta + \phi) \cos(\theta - \phi) \\
 &= \frac{a^2 - b^2}{a^2 + b^2} (a^2 + b^2 - 2),
 \end{aligned}$$

as in the preceding example.

$$\begin{aligned}
 11. \quad \tan \theta + \tan \phi &= \frac{\sin (\theta + \phi)}{\cos \theta \cos \phi} \\
 &= \frac{2ab}{a^2 + b^2} \div \frac{(a^2 + b^2)^2 - 4b^2}{4(a^2 + b^2)}.
 \end{aligned}$$

[See Ex. 4, p. 299 and Ex. 9 above.]

$$12. \quad \tan \frac{\theta}{2} + \tan \frac{\phi}{2} = \frac{\sin \frac{\theta + \phi}{2}}{\cos \frac{\theta}{2} \cos \frac{\phi}{2}} = \frac{2 \sin \frac{\theta + \phi}{2}}{\cos \frac{\theta + \phi}{2} + \cos \frac{\theta - \phi}{2}}.$$

On multiplying numerator and denominator by  $2 \cos \frac{\theta + \phi}{2}$ , this last fraction becomes

$$\frac{2 \sin (\theta + \phi)}{1 + \cos (\theta + \phi) + \cos \theta + \cos \phi}.$$

By substituting for  $\sin (\theta + \phi)$  and  $\cos (\theta + \phi)$  the values found in Ex. 4, p. 299, we obtain

$$\frac{4ab}{a^2 + b^2} \div \left( 1 + \frac{a^2 - b^2}{a^2 + b^2} + a \right),$$

which reduces to

$$\frac{4b}{a^2 + b^2 + 2a}.$$

$$\begin{aligned}
 13. \quad \text{The expression} &= \sin^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma \\
 &= -\cos (\alpha + \beta) \cos (\alpha - \beta) - \cos^2 \gamma + \{\cos (\alpha + \beta) + \cos (\alpha - \beta)\} \cos \gamma \\
 &= -\{\cos (\alpha + \beta) - \cos \gamma\} \{\cos (\alpha - \beta) - \cos \gamma\} \\
 &= 4 \sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{\beta + \gamma - \alpha}{2} \sin \frac{\gamma + \alpha - \beta}{2};
 \end{aligned}$$

from which the second part of the question easily follows.

$$\begin{aligned}
 14. \quad \text{The expression} &= \sin^2 \alpha + (\sin^2 \beta - \sin^2 \gamma) + 2 \sin \alpha \sin \beta \cos \gamma \\
 &= \sin^2 \alpha + \sin (\beta + \gamma) \sin (\beta - \gamma) + \sin \alpha \{\sin (\beta + \gamma) + \sin (\beta - \gamma)\} \\
 &= \{\sin \alpha + \sin (\beta + \gamma)\} \{\sin \alpha + \sin (\beta - \gamma)\} \\
 &= 4 \sin \frac{\alpha + \beta + \gamma}{2} \cos \frac{\beta + \gamma - \alpha}{2} \sin \frac{\alpha + \beta - \gamma}{2} \cos \frac{\alpha - \beta + \gamma}{2}.
 \end{aligned}$$

$$\begin{aligned}
15. \quad \text{The expression} &= \sin^2 \alpha - (\cos^2 \beta - \sin^2 \gamma) - 2 \sin \alpha \sin \beta \sin \gamma \\
&= \sin^2 \alpha - \cos (\beta + \gamma) \cos (\beta - \gamma) - \sin \alpha \{ \cos (\beta - \gamma) - \cos (\beta + \gamma) \} \\
&= \{ \sin \alpha + \cos (\beta + \gamma) \} \{ \sin \alpha - \cos (\beta - \gamma) \} \\
&= - \left\{ \cos \left( \frac{\pi}{2} - \alpha \right) + \cos (\beta + \gamma) \right\} \left\{ \cos \left( \frac{\pi}{2} + \alpha \right) + \cos (\beta - \gamma) \right\} \\
&= - 4 \cos \left( \frac{\beta + \gamma - \alpha}{2} + \frac{\pi}{4} \right) \cos \left( \frac{\beta + \gamma + \alpha}{2} - \frac{\pi}{4} \right) \cos \left( \frac{\alpha + \beta - \gamma}{2} + \frac{\pi}{4} \right) \\
&\qquad \qquad \qquad \cos \left( \frac{\alpha - \beta + \gamma}{2} + \frac{\pi}{4} \right)
\end{aligned}$$

$$\begin{aligned}
16. \quad \tan^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - \cos \alpha + \cos \beta - \cos \alpha \cos \beta}{1 + \cos \alpha - \cos \beta - \cos \alpha \cos \beta} \\
&= \frac{(1 - \cos \alpha)(1 + \cos \beta)}{(1 + \cos \alpha)(1 - \cos \beta)} = \tan^2 \frac{\alpha}{2} \cot^2 \frac{\beta}{2}.
\end{aligned}$$

$$17. \quad \tan^2 \theta = \frac{2 \sin \alpha \sin \beta}{1 + \cos (\alpha + \beta)} = \frac{2 \sin \alpha \sin \beta}{1 + \cos \alpha \cos \beta - \sin \alpha \sin \beta};$$

$$\therefore 1 + \tan^2 \theta = \frac{1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta}{1 + \cos \alpha \cos \beta - \sin \alpha \sin \beta};$$

$$\text{that is,} \quad \sec^2 \theta = \frac{1 + \cos (\alpha - \beta)}{1 + \cos (\alpha + \beta)} = \frac{\cos^2 \frac{\alpha - \beta}{2}}{\cos^2 \frac{\alpha + \beta}{2}}.$$

Taking the positive value of the square root, we have

$$\cos \theta = \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}.$$

$$\therefore \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}}.$$

18. Here  $\tan \theta = \frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta};$

$$\begin{aligned} \therefore \sec^2 \theta &= 1 + \frac{\sin^2 \alpha \cos^2 \beta}{(\cos \alpha + \sin \beta)^2} = \frac{(\cos \alpha + \sin \beta)^2 + (1 - \cos^2 \alpha)(1 - \sin^2 \beta)}{(\cos \alpha + \sin \beta)^2} \\ &= \frac{1 + 2 \cos \alpha \sin \beta + \cos^2 \alpha \sin^2 \beta}{(\cos \alpha + \sin \beta)^2}. \end{aligned}$$

Taking the positive value of the square root, we have

$$\begin{aligned} \cos \theta &= \frac{\cos \alpha + \sin \beta}{1 + \cos \alpha \sin \beta}, \\ \therefore \frac{1 - \cos \theta}{1 + \cos \theta} &= \frac{(1 - \cos \alpha)(1 - \sin \beta)}{(1 + \cos \alpha)(1 + \sin \beta)} \\ &= \frac{(1 - \cos \alpha) \left\{ 1 - \cos \left( \frac{\pi}{2} - \beta \right) \right\}}{(1 + \cos \alpha) \left\{ 1 + \cos \left( \frac{\pi}{2} - \beta \right) \right\}}; \\ \therefore \tan^2 \frac{\theta}{2} &= \tan^2 \frac{\alpha}{2} \tan^2 \left( \frac{\pi}{4} - \frac{\beta}{2} \right). \end{aligned}$$

19. This identity will be established if we shew that

$$\Sigma \sin^3 A \sin B \sin C = 2 \sin A \sin B \sin C (1 + \cos A \cos B \cos C),$$

that is, if  $\Sigma \sin^2 A = 2 (1 + \cos A \cos B \cos C).$

$$\begin{aligned} \text{Now } \sin^2 A + \sin^2 B + \sin^2 C &= \frac{1}{2} (3 - \cos 2A - \cos 2B - \cos 2C) \\ &= \frac{1}{2} (4 + 4 \cos A \cos B \cos C). \end{aligned}$$

[See XII. d. Ex. 9.]

20. Expressing  $a, b, c$  in terms of  $R$ , this identity will be proved if we shew that

$$\Sigma \sin A \cos^3 A = \Pi \sin A (1 - 4 \cos A \cos B \cos C).$$

$$\begin{aligned} \text{Now } 8 \Sigma \sin A \cos^3 A &= 4 \Sigma \cos^2 A \sin 2A \\ &= 2 \Sigma (1 + \cos 2A) \sin 2A \\ &= 2 \Sigma \sin 2A + \Sigma \sin 4A. \end{aligned}$$

$$\text{Now } \Sigma \sin 2A = 4 \Pi \sin A,$$

and

$$\Sigma \sin 4A = -4 \Pi \sin 2A; \quad [\text{Ex. 7, p. 301};]$$

$$\begin{aligned} \therefore \Sigma \sin A \cos^3 A &= \Pi \sin A - 4 \Pi \sin A \cos A \\ &= \Pi \sin A (1 - 4 \cos A \cos B \cos C). \end{aligned}$$

$$\begin{aligned}
 21. \quad \Sigma a^3 \cos (B - C) &= 2R \Sigma a^2 \sin A \cos (B - C) \\
 &= 2R \Sigma a^2 \sin (B + C) \cos (B - C) \\
 &= R \Sigma a^2 (\sin 2B + \sin 2C).
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } a^2 \sin 2B + b^2 \sin 2A &= 2a \sin B \cdot a \cos B + 2b \sin A \cdot b \cos A \\
 &= 2a \sin B (a \cos B + b \cos A) \\
 &= 2ac \sin B = 4\Delta.
 \end{aligned}$$

$$\text{Hence } \Sigma a^3 \cos (B - C) = 12R\Delta = 3abc.$$

22. (1) We have, from the given equation,

$$(b \sin \theta - c)^2 = a^2 \cos^2 \theta = a^2 (1 - \sin^2 \theta),$$

or

$$(a^2 + b^2) \sin^2 \theta - 2bc \sin \theta + (c^2 - a^2) = 0;$$

and this is the required equation since by hypothesis it is satisfied by  $\sin \alpha$  and  $\sin \beta$ .

(2) The required equation is

$$x^2 - (\cos 2\alpha + \cos 2\beta)x + \cos 2\alpha \cos 2\beta = 0.$$

$$\begin{aligned}
 \text{Now } \cos 2\alpha + \cos 2\beta &= 2 \cos (\alpha + \beta) \cos (\alpha - \beta) \\
 &= \frac{2(a^2 - b^2)}{a^2 + b^2} \cdot \frac{2c^2 - a^2 - b^2}{a^2 + b^2}.
 \end{aligned}$$

[See solutions to examples 4 and 6 above.]

$$\begin{aligned}
 \text{Again, } \cos 2\alpha \cos 2\beta &= \cos^2 (\alpha - \beta) - \sin^2 (\alpha + \beta) \\
 &= \frac{(2c^2 - a^2 - b^2)^2 - 4a^2b^2}{(a^2 + b^2)^2}.
 \end{aligned}$$

Hence, by substitution the required equation is obtained.

Otherwise. Let  $\cos \alpha = y$ , then the given equation may be written

$$(ay - c)^2 = b^2(1 - y^2),$$

or

$$(a^2 + b^2)y^2 - b^2 + c^2 = 2acy \dots \dots \dots (1).$$

If

$$x = \cos 2\alpha = 2 \cos^2 \alpha - 1 = 2y^2 - 1,$$

we have

$$y^2 = \frac{x+1}{2}.$$

Substituting in (1) we obtain the equation

$$\left\{ (a^2 + b^2) \frac{x+1}{2} - b^2 + c^2 \right\}^2 = 4a^2c^2 \cdot \left( \frac{x+1}{2} \right),$$

which reduces to

$$\begin{aligned}
 (a^2 + b^2)^2 x^2 - 2(a^2 - b^2)(2c^2 - a^2 - b^2)x \\
 + (a^4 + b^4 + 4c^4 - 2a^2b^2 - 4a^2c^2 - 4b^2c^2) = 0.
 \end{aligned}$$

**EXAMPLES. XXIV. b. PAGE 307.**

$$\begin{aligned}
 1. \quad \Sigma \sin (a - \theta) \sin (\beta - \gamma) &= \Sigma (\sin a \cos \theta - \cos a \sin \theta) \sin (\beta - \gamma) \\
 &= \cos \theta \Sigma \sin a \sin (\beta - \gamma) - \sin \theta \Sigma \cos a \sin (\beta - \gamma) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \Sigma (\cos \beta \cos \gamma - \sin \beta \sin \gamma) \sin (\beta - \gamma) &= \Sigma \cos (\beta + \gamma) \sin (\beta - \gamma) \\
 &= \frac{1}{2} \Sigma (\sin 2\beta - \sin 2\gamma) \\
 &= 0. \\
 \therefore \Sigma \cos \beta \cos \gamma \sin (\beta - \gamma) &= \Sigma \sin \beta \sin \gamma \sin (\beta - \gamma).
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \Sigma \sin (\beta - \gamma) \cos (\beta + \gamma + \theta) \\
 &= \Sigma \sin (\beta - \gamma) \{ \cos \theta \cos (\beta + \gamma) - \sin \theta \sin (\beta + \gamma) \} \\
 &= \cos \theta \Sigma \sin (\beta - \gamma) \cos (\beta + \gamma) - \sin \theta \Sigma \sin (\beta - \gamma) \sin (\beta + \gamma) \\
 &= \frac{1}{2} \cos \theta \Sigma (\sin 2\beta - \sin 2\gamma) - \frac{1}{2} \sin \theta \Sigma (\cos 2\gamma - \cos 2\beta) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \cos 2 (\beta - \gamma) + \cos 2 (\gamma - \alpha) + \cos 2 (\alpha - \beta) \\
 &= 2 \cos (\beta - \alpha) \cos (\alpha + \beta - 2\gamma) + 2 \cos^2 (\alpha - \beta) - 1 \\
 &= 2 \cos (\alpha - \beta) \{ \cos (\alpha + \beta - 2\gamma) + \cos (\alpha - \beta) \} - 1 \\
 &= 4 \cos (\alpha - \beta) \cos (\alpha - \gamma) \cos (\beta - \gamma) - 1.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \Sigma \sin \beta \sin \gamma \sin (\beta - \gamma) \\
 &= \frac{1}{2} \Sigma \{ \cos (\beta - \gamma) - \cos (\beta + \gamma) \} \sin (\beta - \gamma) \\
 &= \frac{1}{4} \Sigma \sin 2 (\beta - \gamma) - \frac{1}{4} \Sigma (\sin 2\beta - \sin 2\gamma) \\
 &= -\Pi \sin (\beta - \gamma).
 \end{aligned}$$

[Art. 306, Ex. 2.]

$$\begin{aligned}
 6. \quad \cot (\alpha - \beta) &= \cot \{ (\alpha - \gamma) - (\beta - \gamma) \} \\
 &= \frac{\cot (\alpha - \gamma) \cot (\beta - \gamma) + 1}{\cot (\beta - \gamma) - \cot (\alpha - \gamma)};
 \end{aligned}$$

by multiplying up and transposing we obtain the required result.

$$7. \quad 2\Sigma \sin 3\alpha \sin (\beta - \gamma) = \cos (3\alpha - \beta + \gamma) - \cos (3\alpha + \beta - \gamma) + \cos (3\beta - \gamma + \alpha) \\ - \cos (3\beta + \gamma - \alpha) + \cos (3\gamma - \alpha + \beta) - \cos (3\gamma + \alpha - \beta).$$

Combining the first and fourth terms, the second and fifth terms, the third and sixth terms, and dividing by 2, we obtain

$$\Sigma \sin 3\alpha \sin (\beta - \gamma) = \sin (\alpha + \beta + \gamma) \{ \sin 2(\beta - \alpha) + \sin 2(\alpha - \gamma) + \sin 2(\gamma - \beta) \} \\ = 4 \sin (\alpha + \beta + \gamma) \Pi \sin (\beta - \gamma). \quad [\text{Art. 306, Ex. 2.}]$$

$$8. \quad 4\Sigma \cos^3 \alpha \sin (\beta - \gamma) = \Sigma (\cos 3\alpha + 3 \cos \alpha) \sin (\beta - \gamma) \\ = \Sigma \cos 3\alpha \sin (\beta - \gamma) \\ = 4 \cos (\alpha + \beta + \gamma) \Pi \sin (\beta - \gamma). \quad [\text{Art. 306, Ex. 4.}]$$

$$9. \quad 4\Sigma \cos (\theta + \alpha) \sin (\theta - \alpha) \cos (\beta + \gamma) \sin (\beta - \gamma) \\ = \Sigma (\sin 2\theta - \sin 2\alpha) (\sin 2\beta - \sin 2\gamma) \\ = \sin 2\theta \Sigma (\sin 2\beta - \sin 2\gamma) - \Sigma \sin 2\alpha (\sin 2\beta - \sin 2\gamma) \\ = 0.$$

$$10. \quad \text{In the identity } \Sigma bc(b-c) = -\Pi(b-c), \text{ put} \\ a = \sin^2 \alpha, \quad b = \sin^2 \beta, \quad c = \sin^2 \gamma;$$

then  $b - c = \sin^2 \beta - \sin^2 \gamma = \sin (\beta + \gamma) \sin (\beta - \gamma);$

$$\therefore \Sigma \sin^2 \beta \sin^2 \gamma \sin (\beta + \gamma) \sin (\beta - \gamma) = -\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

$$11. \quad \text{In the identity } \Sigma bc(b-c) = -\Pi(b-c), \text{ put}$$

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

then  $b - c = \cos 2\beta - \cos 2\gamma = -2 \sin (\beta + \gamma) \sin (\beta - \gamma);$

$$\therefore -2\Sigma \cos 2\beta \cos 2\gamma \sin (\beta + \gamma) \sin (\beta - \gamma) = 8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

$$12. \quad \text{In the identity } \Sigma a^2(b-c) = -\Pi(b-c), \text{ put}$$

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

then  $b - c = \cos 2\beta - \cos 2\gamma = -2 \sin (\beta + \gamma) \sin (\beta - \gamma);$

$$\therefore \Sigma 2 \cos^2 2\alpha \sin (\beta + \gamma) \sin (\beta - \gamma) = -8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

Also  $\Sigma \sin (\beta + \gamma) \sin (\beta - \gamma) = 0;$

whence by subtraction, we have

$$\Sigma (2 \cos^2 2\alpha - 1) \sin (\beta + \gamma) \sin (\beta - \gamma) = 8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

13. In the identity  $\Sigma a^3(b-c) = -(a+b+c)\Pi(b-c)$ , put

$$a = \sin \alpha, \quad b = \sin \beta, \quad c = \sin \gamma;$$

$$\therefore \Sigma \sin^3 \alpha (\sin \beta - \sin \gamma) = -(\sin \alpha + \sin \beta + \sin \gamma) \Pi (\sin \beta - \sin \gamma) \dots (1).$$

But  $\Sigma \sin \alpha (\sin \beta - \sin \gamma) = 0 \dots \dots \dots (2);$

multiply (2) by 3, and (1) by 4; then by subtraction we obtain

$$\Sigma (3 \sin \alpha - 4 \sin^3 \alpha) (\sin \beta - \sin \gamma) = 4 (\sin \alpha + \sin \beta + \sin \gamma) \Pi (\sin \beta - \sin \gamma).$$

14. If  $a+b+c=0$ , then  $a^3+b^3+c^3=3abc$ .

The condition  $a+b+c=0$  is satisfied, if  $a = \sin(\beta+\gamma) \sin(\beta-\gamma)$ , and  $b$  and  $c$  are equal to corresponding quantities.

15. The condition  $a+b+c=0$  is satisfied, if  $a = \cos(\beta+\gamma+\theta) \sin(\beta-\gamma)$ , and  $b$  and  $c$  are equal to corresponding quantities.

16. We proceed exactly as in Art. 309, and shew that

$$\alpha + \beta + \gamma = n\pi;$$

$$\therefore 3\alpha + 3\beta + 3\gamma = 3n\pi.$$

From this relation it is easy to shew that

$$\Sigma \tan 3\alpha = \Pi \tan 3\alpha;$$

$$\therefore \Sigma \frac{3x - x^3}{1 - 3x^2} = \Pi \frac{3x - x^3}{1 - 3x^2}.$$

17. Put  $x = \cot \alpha$ ,  $y = \cot \beta$ ,  $z = \cot \gamma$ ; then

$$\cot \beta \cot \gamma + \cot \gamma \cot \alpha + \cot \alpha \cot \beta = 1;$$

$$\therefore \cot \alpha = -\frac{\cot \beta \cot \gamma - 1}{\cot \gamma + \cot \beta} = -\cot(\beta + \gamma);$$

$$\therefore \alpha = n\pi - (\beta + \gamma), \text{ or } \alpha + \beta + \gamma = n\pi;$$

$$\therefore 2\alpha + 2\beta + 2\gamma = 2n\pi.$$

From this relation it is easy to shew that

$$\cot 2\beta \cot 2\gamma + \cot 2\gamma \cot 2\alpha + \cot 2\alpha \cot 2\beta = 1;$$

$$\therefore \frac{(y^2 - 1)(z^2 - 1)}{4yz} + \frac{(z^2 - 1)(x^2 - 1)}{4zx} + \frac{(x^2 - 1)(y^2 - 1)}{4xy} = 1,$$

$$\therefore \Sigma x(1 - y^2)(1 - z^2) = 4xyz.$$

## EXAMPLES. XXIV. c. PAGE 311.

1. If  $A + B + C = 0$ ,

then  $\cot C = -\cot(A + B) = -\frac{\cot A \cot B - 1}{\cot B + \cot A}$ ;

that is,  $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$ .

The given condition is satisfied if

$$A = 2\beta + \gamma - 3\alpha, \quad B = 2\gamma + \alpha - 3\beta, \quad C = 2\alpha + \beta - 3\gamma.$$

2. (1) In Example 4, Art. 133, we have proved that

$$4 \sin \alpha \sin \beta \sin \gamma = \Sigma \sin(\beta + \gamma - \alpha) - \sin(\alpha + \beta + \gamma).$$

In this identity first replace  $\alpha, \beta, \gamma$  by  $\beta + \gamma, \gamma + \alpha, \alpha + \beta$  respectively, and secondly replace  $\alpha, \beta, \gamma$  by  $2\alpha, 2\beta, 2\gamma$  respectively.

Thus  $8\Pi \sin(\beta + \gamma) = 2 \sin 2\alpha - 2 \sin 2(\alpha + \beta + \gamma)$ ,

and  $4\Pi \sin 2\alpha = \Sigma \sin 2(\beta + \gamma - \alpha) - \sin 2(\alpha + \beta + \gamma)$ .

$$\begin{aligned} \therefore 8\Pi \sin(\beta + \gamma) + 4\Pi \sin 2\alpha &= 2\Sigma \sin 2\alpha + \Sigma \sin 2(\beta + \gamma - \alpha) - 3 \sin 2(\alpha + \beta + \gamma) \\ &= 2\Sigma \sin 2\alpha + \Sigma \{ \sin 2(\beta + \gamma - \alpha) - \sin 2(\alpha + \beta + \gamma) \} \\ &= 2\Sigma \sin 2\alpha - 2\Sigma \cos 2(\beta + \gamma) \sin 2\alpha \\ &= 2\Sigma \sin 2\alpha \{ 1 - \cos 2(\beta + \gamma) \} \\ &= 4\Sigma \sin 2\alpha \sin^2(\beta + \gamma). \end{aligned}$$

(2) In the first part of this Example, we have seen that

$$8\Pi \sin \alpha = 2\Sigma \sin(\beta + \gamma - \alpha) - 2 \sin(\alpha + \beta + \gamma).$$

By replacing  $\alpha, \beta, \gamma$  by  $\beta + \gamma - \alpha, \gamma + \alpha - \beta, \alpha + \beta - \gamma$  respectively, we have

$$4\Pi \sin(\beta + \gamma - \alpha) = \Sigma \sin(3\alpha - \beta - \gamma) - \sin(\alpha + \beta + \gamma).$$

$$\begin{aligned} \therefore 4\Pi \sin(\beta + \gamma - \alpha) + 8\Pi \sin \alpha &= 2\Sigma \sin(\beta + \gamma - \alpha) + \Sigma \sin(3\alpha - \beta - \gamma) - 3 \sin(\alpha + \beta + \gamma) \\ &= 2\Sigma \sin(\beta + \gamma - \alpha) + \Sigma \{ \sin(3\alpha - \beta - \gamma) - \sin(\alpha + \beta + \gamma) \} \\ &= 2\Sigma \sin(\beta + \gamma - \alpha) - 2\Sigma \cos 2\alpha \sin(\beta + \gamma - \alpha) \\ &= 2\Sigma \sin(\beta + \gamma - \alpha) \{ 1 - \cos 2\alpha \} \\ &= 4\Sigma \sin^2 \alpha \sin(\beta + \gamma - \alpha). \end{aligned}$$

3. (1) This is equivalent to proving that in the pedal triangle

$$a'^2 - b'^2 = 2R'c' \sin(A' - B').$$

Now in *any* triangle

$$\begin{aligned} a^2 - b^2 &= 4R^2(\sin^2 A - \sin^2 B) \\ &= 4R^2 \sin(A+B) \sin(A-B) \\ &= 2R \cdot 2R \sin C \cdot \sin(A-B) \\ &= 2Rc \sin(A-B). \end{aligned}$$

(2) This is equivalent to proving that in the ex-central triangle

$$a_1^2 - b_1^2 = 2R_1c_1 \sin(A_1 - B_1).$$

This identity has been proved in (1).

(3) This is equivalent to proving that in the pedal triangle

$$\Sigma(b' + c') \tan \frac{A'}{2} = 4R' \Sigma \cos A'.$$

Now in *any* triangle

$$\begin{aligned} \Sigma(b+c) \tan \frac{A}{2} &= 2R \Sigma(\sin B + \sin C) \tan \frac{A}{2} \\ &= 4R \Sigma \sin \frac{B+C}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \div \cos \frac{A}{2} \\ &= 4R \Sigma \cos \frac{B-C}{2} \cos \frac{B+C}{2} \\ &= 2R \Sigma(\cos B + \cos C) \\ &= 4R(\cos A + \cos B + \cos C). \end{aligned}$$

4. We have  $\sin^2 2\theta = 4 \sin^2 \alpha \sin^2 \gamma = (1 - \cos 2\alpha)(1 - \cos 2\gamma)$ ;

$$\therefore 1 - \cos^2 2\theta = \left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right) \left(1 - \frac{\cos 2\theta}{\cos 2\delta}\right);$$

$$\therefore \cos 2\theta \left(\frac{1}{\cos 2\beta} + \frac{1}{\cos 2\delta}\right) = \cos^2 2\theta \left(1 + \frac{1}{\cos 2\beta \cos 2\delta}\right);$$

$$\therefore \cos 2\theta = \frac{\cos 2\beta + \cos 2\delta}{1 + \cos 2\beta \cos 2\delta};$$

$$\therefore \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{(1 - \cos 2\beta)(1 - \cos 2\delta)}{(1 + \cos 2\beta)(1 + \cos 2\delta)};$$

$$\therefore \tan^2 \theta = \tan^2 \beta \tan^2 \delta.$$

5. We have

$$\tan^2 \frac{\gamma}{2} = \tan^2 \frac{\theta}{2} \tan^2 \frac{\phi}{2};$$

$$\begin{aligned} \therefore \frac{1 - \cos \gamma}{1 + \cos \gamma} &= \frac{1 - \cos \theta}{1 + \cos \theta} \cdot \frac{1 - \cos \phi}{1 + \cos \phi} \\ &= \frac{1 - \cos \alpha \cos \gamma}{1 + \cos \alpha \cos \gamma} \cdot \frac{1 - \cos \beta \cos \gamma}{1 + \cos \beta \cos \gamma}. \end{aligned}$$

Componendo and Dividendo,

$$\frac{1}{\cos \gamma} = \frac{1 + \cos \alpha \cos \beta \cos^2 \gamma}{(\cos \alpha + \cos \beta) \cos \gamma};$$

$$\therefore \cos^2 \gamma = \frac{\cos \alpha + \cos \beta - 1}{\cos \alpha \cos \beta};$$

$$\therefore \sin^2 \gamma = \frac{(1 - \cos \alpha)(1 - \cos \beta)}{\cos \alpha \cos \beta} = (\sec \alpha - 1)(\sec \beta - 1).$$

6. By solving for  $\cos \theta$ , we have

$$\cos \theta = \frac{\sin^2 \beta \cos^2 \alpha - \sin^2 \alpha \cos^2 \beta}{\sin^2 \beta \cos \alpha - \sin^2 \alpha \cos \beta}.$$

By substituting  $1 - \cos^2 \beta$  for  $\sin^2 \beta$ , and  $1 - \cos^2 \alpha$  for  $\sin^2 \alpha$ , we have

$$\begin{aligned} \cos \theta &= \frac{\cos^2 \alpha - \cos^2 \beta}{(1 - \cos^2 \beta) \cos \alpha - (1 - \cos^2 \alpha) \cos \beta} \\ &= \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta}. \end{aligned}$$

$$\therefore \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{(1 - \cos \alpha)(1 - \cos \beta)}{(1 + \cos \alpha)(1 + \cos \beta)};$$

$$\therefore \tan^2 \frac{\theta}{2} = \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2}.$$

7. From the two given equations, we have

$$\sec \alpha = \frac{\tan \beta}{\tan \theta}, \text{ and } \tan \alpha = \frac{\tan \gamma}{\sin \theta};$$

$$\therefore \frac{\tan^2 \beta}{\tan^2 \theta} - \frac{\tan^2 \gamma}{\sin^2 \theta} = 1;$$

$$\therefore \frac{\tan^2 \beta \cos^2 \theta}{1 - \cos^2 \theta} - \frac{\tan^2 \gamma}{1 - \cos^2 \theta} = 1;$$

$$\therefore \tan^2 \beta \cos^2 \theta - \tan^2 \gamma = 1 - \cos^2 \theta;$$

$$\therefore \cos^2 \theta = \frac{1 + \tan^2 \gamma}{1 + \tan^2 \beta} = \frac{\sec^2 \gamma}{\sec^2 \beta}.$$

8. We have  $bc \cos \alpha \cos \phi + ac \sin \alpha \sin \phi - ab = 0$ ,  
and  $bc \cos \beta \cos \phi + ac \sin \beta \sin \phi - ab = 0$ ;

whence by cross multiplication

$$\frac{\cos \phi}{a^2 bc (\sin \beta - \sin \alpha)} = \frac{\sin \phi}{ab^2 c (\cos \alpha - \cos \beta)} = \frac{1}{abc^2 \sin (\beta - \alpha)};$$

$$\therefore \frac{\cos \phi}{a \cos \frac{\alpha + \beta}{2}} = \frac{\sin \phi}{b \sin \frac{\alpha + \beta}{2}} = \frac{1}{c \cos \frac{\alpha - \beta}{2}};$$

$$\therefore a^2 \cos^2 \frac{\alpha + \beta}{2} + b^2 \sin^2 \frac{\alpha + \beta}{2} = c^2 \cos^2 \frac{\alpha - \beta}{2};$$

$$\therefore a^2 \{1 + \cos (\alpha + \beta)\} + b^2 \{1 - \cos (\alpha + \beta)\} = c^2 \{1 + \cos (\alpha - \beta)\};$$

$$\therefore (b^2 + c^2 - a^2) \cos \alpha \cos \beta + (c^2 + a^2 - b^2) \sin \alpha \sin \beta = a^2 + b^2 - c^2.$$

9. From the given equation, we have

$$\sin^2 \alpha (\cos \theta - \cos \alpha)^2 = \cos^2 \alpha \sin^2 \theta = \cos^2 \alpha (1 - \cos^2 \theta);$$

$$\therefore \cos^2 \theta - 2 \cos \alpha \sin^2 \alpha \cos \theta - \cos^4 \alpha = 0;$$

which is a quadratic in  $\cos \theta$  with roots  $\cos \beta$  and  $\cos \gamma$ .

$$\therefore \cos \beta \cos \gamma = \cos^4 \alpha.$$

Similarly, from the equation

$$\cos^2 \alpha (\sin \theta - \sin \alpha)^2 = \sin^2 \alpha \cos^2 \theta = \sin^2 \alpha (1 - \sin^2 \theta),$$

we may shew that  $\sin \beta \sin \gamma = \sin^4 \alpha$ .

$$\therefore \frac{\cos \beta \cos \gamma}{\cos^2 \alpha} + \frac{\sin \beta \sin \gamma}{\sin^2 \alpha} = \cos^2 \alpha + \sin^2 \alpha = 1.$$

10. We have  $k^2 \cos \beta \cos \alpha + k \sin \alpha + (k \sin \beta + 1) = 0$  .....(1),  
and  $k^2 \cos \gamma \cos \alpha + k \sin \alpha + (k \sin \gamma + 1) = 0$  .....(2);

whence by cross multiplication,

$$\frac{\cos \alpha}{k^2 (\sin \beta - \sin \gamma)} = \frac{\sin \alpha}{k^3 \sin (\beta - \gamma) + k^2 (\cos \gamma - \cos \beta)} = \frac{1}{k^3 (\cos \beta - \cos \gamma)};$$

$$\therefore \frac{\cos \alpha}{\cos \frac{\beta + \gamma}{2}} = \frac{\sin \alpha}{k \cos \frac{\beta - \gamma}{2} + \sin \frac{\beta + \gamma}{2}} = - \frac{1}{k \sin \frac{\beta + \gamma}{2}};$$

$$\therefore \cos^2 \frac{\beta + \gamma}{2} + \left\{ k \cos \frac{\beta - \gamma}{2} + \sin \frac{\beta + \gamma}{2} \right\}^2 = k^2 \sin^2 \frac{\beta + \gamma}{2};$$

$$\therefore k^2 \left( \cos^2 \frac{\beta - \gamma}{2} - \sin^2 \frac{\beta + \gamma}{2} \right) + k (\sin \beta + \sin \gamma) + 1 = 0;$$

$$\therefore k^2 \cos \beta \cos \gamma + k (\sin \beta + \sin \gamma) + 1 = 0.$$

Otherwise. Form a quadratic equation in  $\sin \theta$ ; thus

$$k^4 \cos^2 a (1 - \sin^2 \theta) = (1 + k \sin a + k \sin \theta)^2;$$

$\sin \beta + \sin \gamma =$  sum of roots of this equation

$$= - \frac{\text{coefficient of } \sin \theta}{\text{coefficient of } \sin^2 \theta}$$

$$= - \frac{2(1 + k \sin a)k}{k^2(1 + k^2 \cos^2 a)};$$

$$\therefore k(\sin \beta + \sin \gamma) = - \frac{2(1 + k \sin a)}{1 + k^2 \cos^2 a} \dots \dots \dots (1).$$

Again, form a quadratic equation in  $\cos \theta$ ; thus

$$k^2(1 - \cos^2 \theta) = (1 + k \sin a + k^2 \cos a \cos \theta)^2;$$

$\cos \beta \cos \gamma =$  product of roots of this equation

$$= \frac{k^2 - (1 + k \sin a)^2}{-k^2 - k^4 \cos^2 a};$$

$$\therefore k^2 \cos \beta \cos \gamma = \frac{1 + 2k \sin a - k^2 \cos^2 a}{1 + k^2 \cos^2 a} \dots \dots \dots (2).$$

By adding (1) and (2) we obtain the required result. •

11. We have

$$\sin^2 a \cos \beta \cos \phi + \cos^2 a \sin \beta \sin \phi + \cos^2 a \sin^2 a = 0,$$

and

$$\sin^2 a \cos \gamma \cos \phi + \cos^2 a \sin \gamma \sin \phi + \cos^2 a \sin^2 a = 0;$$

whence by cross multiplication

$$\frac{\cos \phi}{\cos^4 a \sin^2 a (\sin \beta - \sin \gamma)} = \frac{\sin \phi}{\cos^2 a \sin^4 a (\cos \gamma - \cos \beta)}$$

$$= \frac{1}{\sin^2 a \cos^2 a \sin(\gamma - \beta)};$$

$$\therefore \frac{\cos \phi}{\cos^2 a \cos^{\frac{\beta+\gamma}{2}}} = \frac{\sin \phi}{\sin^2 a \sin^{\frac{\beta+\gamma}{2}}} = - \frac{1}{\cos^{\frac{\beta-\gamma}{2}}};$$

$$\therefore \cos^4 a \cos^2 \frac{\beta+\gamma}{2} + \sin^4 a \sin^2 \frac{\beta+\gamma}{2} = \cos^2 \frac{\beta-\gamma}{2};$$

$$\therefore \cos^4 a \{1 + \cos(\beta + \gamma)\} + \sin^4 a \{1 - \cos(\beta + \gamma)\} = 1 + \cos(\beta - \gamma);$$

$$\therefore \cos \beta \cos \gamma (1 - \cos^4 a + \sin^4 a) + \sin \beta \sin \gamma (1 + \cos^4 a - \sin^4 a) = \cos^4 a + \sin^4 a - 1.$$

Writing  $\cos^4 a + \sin^4 a + 2 \sin^2 a \cos^2 a$  instead of unity we have

$$2 \cos \beta \cos \gamma \sin^2 a (\cos^2 a + \sin^2 a) + 2 \sin \beta \sin \gamma \cos^2 a (\cos^2 a + \sin^2 a) = -2 \sin^2 a \cos^2 a;$$

that is,  $\sin^2 a \cos \beta \cos \gamma + \cos^2 a \sin \beta \sin \gamma + \sin^2 a \cos^2 a = 0.$

**EXAMPLES. XXV. a. PAGE 318.**

$$1. \quad p \cot \theta + q \tan \theta = (\sqrt{p \cot \theta} - \sqrt{q \tan \theta})^2 + 2 \sqrt{pq}.$$

$$2. \quad 4 \sin^2 \theta + \operatorname{cosec}^2 \theta = (2 \sin \theta - \operatorname{cosec} \theta)^2 + 4.$$

$$3. \quad 8 \sec^2 \theta + 18 \cos^2 \theta = 2 \{ (2 \sec \theta - 3 \cos \theta)^2 + 12 \}.$$

$$4. \quad 3 - 2 \cos \theta + \cos^2 \theta = 2 + (1 - \cos \theta)^2.$$

5. We have  $\tan^2 \beta + \tan^2 \gamma > 2 \tan \beta \tan \gamma$ , and two corresponding inequalities. [See Art. 316.]

$$6. \quad \text{Since } (1 - \sin \alpha)^2 \text{ is positive, } 1 + \sin^2 \alpha > 2 \sin \alpha.$$

$$\text{Similarly,} \quad 1 + \sin^2 \beta > 2 \sin \beta.$$

$$\therefore 2 + \sin^2 \alpha + \sin^2 \beta > 2 \sin \alpha + 2 \sin \beta.$$

$$7. \quad \sin \theta + \cos \theta = \sqrt{2} \sin \left( \theta + \frac{\pi}{4} \right).$$

$$8. \quad \cos \theta + \sqrt{3} \sin \theta = 2 \sin \left( \theta + \frac{\pi}{6} \right).$$

$$9. \quad a \cos (\alpha + \theta) + b \sin \theta = a \cos \alpha \cos \theta + (b - a \sin \alpha) \sin \theta.$$

$$\therefore \text{maximum value} = \sqrt{a^2 \cos^2 \alpha + (b - a \sin \alpha)^2}. \quad [\text{Art. 317.}]$$

$$10. \quad p \cos \theta + q \sin (\alpha + \theta) = (p + q \sin \alpha) \cos \theta + q \cos \alpha \sin \theta.$$

$$\therefore \text{maximum value} = \sqrt{(p + q \sin \alpha)^2 + q^2 \cos^2 \alpha}.$$

$$11. \quad \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} = 2 \sin \frac{\sigma}{2} \cos \frac{\alpha - \beta}{2}.$$

$$\therefore \text{maximum value} = 2 \sin \frac{\sigma}{2}.$$

$$12. \quad \sin \alpha \sin \beta = \frac{1}{2} \{ \cos (\alpha - \beta) - \cos (\alpha + \beta) \} = \frac{1}{2} \{ \cos (\alpha - \beta) - \cos \sigma \}.$$

$$\therefore \text{maximum value} = \frac{1}{2} (1 - \cos \sigma) = \sin^2 \frac{\sigma}{2}.$$

$$13. \quad \tan \alpha + \tan \beta = \frac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta} = \frac{\sin \sigma}{\cos \alpha \cos \beta}.$$

By Art. 319, the denominator is a maximum when  $\alpha = \beta$ , and in this case  $\tan \alpha + \tan \beta$  is a minimum, its value being  $2 \tan \frac{\sigma}{2}$ .

$$\begin{aligned}
 14. \quad \operatorname{cosec} \alpha + \operatorname{cosec} \beta &= \frac{\sin \alpha + \sin \beta}{\sin \alpha \sin \beta} \\
 &= \frac{4 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}}{\cos (\alpha - \beta) - \cos (\alpha + \beta)} = \frac{2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}}{\cos^2 \frac{\alpha - \beta}{2} - \cos^2 \frac{\alpha + \beta}{2}} \\
 &= \sin \frac{\alpha + \beta}{2} \left( \frac{1}{\cos \frac{\alpha - \beta}{2} + \cos \frac{\alpha + \beta}{2}} + \frac{1}{\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2}} \right).
 \end{aligned}$$

Since  $\alpha + \beta$  is constant, this expression is least when the denominators are greatest, that is, when  $\alpha = \beta = \frac{\sigma}{2}$ .

Thus the minimum value is  $2 \operatorname{cosec} \frac{\sigma}{2}$ .

$$\begin{aligned}
 15. \quad \cos A \cos B \cos C &= \frac{1}{2} \cos C \{ \cos (A - B) + \cos (A + B) \} \\
 &= \frac{1}{2} \cos C \{ \cos (A - B) - \cos C \}.
 \end{aligned}$$

Supposing  $C$  constant, this expression is not a maximum unless  $A = B$ . Similarly, we may shew that the given expression is not a maximum unless  $A = B = C = 60^\circ$ . In this case its value is  $\cos^3 60^\circ$  or  $\frac{1}{8}$ .

$$16. \quad \cot A + \cot B + \cot C = \frac{\sin (A + B)}{\sin A \sin B} + \cot C = \frac{\sin C}{\sin A \sin B} + \cot C.$$

Supposing  $C$  constant,  $\sin A \sin B$  is a maximum when  $A = B$ , and in this case the given expression is a minimum. Thus the given expression is a minimum when  $A = B = C$ , and its value is  $3 \cot 60^\circ$  or  $\sqrt{3}$ .

$$\begin{aligned}
 17. \quad \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} &= \frac{1}{2} (1 - \cos A) + \dots + \\
 &= \frac{3}{2} - \frac{1}{2} (\cos A + \cos B + \cos C).
 \end{aligned}$$

As in Example 1, p. 315, it is easy to shew that  $\cos A + \cos B + \cos C$  is a maximum when  $A = B = C$ , and in this case the given expression is a minimum, its value being  $3 \sin^2 \frac{60^\circ}{2}$  or  $\frac{3}{4}$ .

18. If  $C$  is constant,  $A + B$  is constant, and therefore  $\sec A + \sec B$  is a minimum when  $A = B$ . [See Ex. 2, p. 316.]

Thus the given expression is a minimum when  $A = B = C$ , its value being  $3 \sec 60^\circ$  or 6.

$$\begin{aligned}
 19. \quad \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \\
 &= \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2 \sum \tan \frac{B}{2} \tan \frac{C}{2} \\
 &= \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2.
 \end{aligned}$$

As in Example 16, it is easy to shew that  $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}$  is a minimum when  $A = B = C$ , and in this case the given expression is a minimum, its value being  $3 \tan^2 \frac{60^\circ}{2}$  or 1.

$$\begin{aligned}
 20. \quad \cot^2 A + \cot^2 B + \cot^2 C &= (\cot A + \cot B + \cot C)^2 - 2 \sum \cot B \cot C \\
 &= (\cot A + \cot B + \cot C)^2 - 2.
 \end{aligned}$$

But it has been shewn in Example 16 that the right side is a minimum when  $A = B = C$ , and in this case the given expression is a minimum, its value being  $3 \cot^2 60^\circ$  or 1.

$$\begin{aligned}
 21. \quad 2(a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta) \\
 &= a(1 - \cos 2\theta) + b \sin 2\theta + c(1 + \cos 2\theta) \\
 &= a + c + b \sin 2\theta - (a - c) \cos 2\theta \dots\dots\dots(1).
 \end{aligned}$$

The greatest value of  $b \sin 2\theta - (a - c) \cos 2\theta$  is  $\sqrt{b^2 + (a - c)^2}$ , which is less than  $\sqrt{4ac + (a - c)^2}$  or  $a + c$ , since  $b^2 < 4ac$ .

Hence as in Art. 317, the maximum and minimum values of (1) are

$$a + c \pm \sqrt{b^2 + (a - c)^2}.$$

$$\begin{aligned}
 22. \quad \sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{\alpha + \beta}{2} \\
 &= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + \alpha}{2} \sin \frac{\beta + \gamma}{2}.
 \end{aligned}$$

The expression on the right is positive, since each of its component factors is positive.

$$\therefore \sin \alpha + \sin \beta + \sin \gamma > \sin (\alpha + \beta + \gamma).$$

23. Let  $x = a \operatorname{cosec} \theta - b \cot \theta$ , and put  $\cot \theta = t$ ;  
then

$$x = a \sqrt{1 + t^2} - bt.$$

As in Ex. 2, page 317, we may shew that  $x > \sqrt{a^2 - b^2}$ ;

$$\therefore a \operatorname{cosec} \theta - b \cot \theta > \sqrt{a^2 - b^2}.$$

24. Let  $x = \frac{\sec^2 \theta - \tan \theta}{\sec^2 \theta + \tan \theta} = \frac{1 - \tan \theta + \tan^2 \theta}{1 + \tan \theta + \tan^2 \theta};$

$$\therefore \tan^2 \theta (x - 1) + \tan \theta (x + 1) + x - 1 = 0.$$

In order that the values of  $\tan \theta$  found from this quadratic may be real, we must have

$$(x + 1)^2 > 4(x - 1)^2;$$

that is,  $-3x^2 + 10x - 3$  must be positive;

that is,  $(3x - 1)(x - 3)$  must be negative.

Hence  $x$  must lie between 3 and  $\frac{1}{3}$ .

25. Denote the expression by  $x$ ; then

$$x = \frac{\tan^4 \theta + \tan^2 \theta - 1}{\tan^4 \theta - \tan^2 \theta + 1};$$

$$\therefore \tan^4 \theta (x - 1) - \tan^2 \theta (x + 1) + x + 1 = 0.$$

In order that the values of  $\tan^2 \theta$  found from this equation may be real, we must have

$$(x + 1)^2 > 4(x + 1)(x - 1);$$

$$\therefore (x + 1)(5 - 3x) > 0.$$

Thus the greatest value of  $x$  is  $\frac{5}{3}$ .

26. We have

$$\begin{aligned} & (a^2 + b^2 + c^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - (a \cos \alpha + b \cos \beta + c \cos \gamma)^2 \\ &= (b \cos \gamma - c \cos \beta)^2 + (c \cos \alpha - a \cos \gamma)^2 + (a \cos \beta - b \cos \alpha)^2, \end{aligned}$$

the minimum value of which is zero.

$$\therefore \text{minimum value of } (a^2 + b^2 + c^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - k^2 = 0.$$

$$\begin{aligned} \text{Again, } & (a + b + c)(a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma) - (a \cos \alpha + b \cos \beta + c \cos \gamma)^2 \\ &= bc(\cos \beta - \cos \gamma)^2 + ca(\cos \gamma - \cos \alpha)^2 + ab(\cos \alpha - \cos \beta)^2 \end{aligned}$$

the minimum value of which is zero.

$$\therefore \text{minimum value of } (a + b + c)(a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma) - k^2 = 0.$$

**EXAMPLES. XXV. b. PAGE 324.**

1. By squaring each of the given equations and adding, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

2. By transposition, we have

$$a \sec \theta = x \tan \theta + y, \quad b \sec \theta = x - y \tan \theta;$$

whence by squaring and adding,

$$(a^2 + b^2) \sec^2 \theta = (x^2 + y^2) (1 + \tan^2 \theta),$$

or

$$x^2 + y^2 = a^2 + b^2.$$

3. We have  $\cos \theta + \sin \theta = a$ , and  $\cos^2 \theta - \sin^2 \theta = b$ ;

$$\therefore \cos \theta - \sin \theta = \frac{b}{a};$$

$$\therefore a^2 + \frac{b^2}{a^2} = 2.$$

4. We have  $x^2 = (\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$ ,

and

$$y = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{2}{\sin 2\theta};$$

$$\therefore y(x^2 - 1) = 2.$$

5. By addition and subtraction, we have

$$\cot \theta = \frac{a+b}{2}, \text{ and } \cos \theta = \frac{a-b}{2};$$

whence by division,

$$\sin \theta = \frac{a-b}{a+b};$$

$$\therefore \left(\frac{a-b}{2}\right)^2 + \left(\frac{a-b}{a+b}\right)^2 = 1, \text{ or } \left(\frac{a-b}{2}\right)^2 = \frac{4ab}{(a+b)^2},$$

that is,

$$(a^2 - b^2)^2 = 16ab.$$

$$6. \text{ Here } x = \cot \theta + \frac{1}{\cot \theta} = \frac{\cot^2 \theta + 1}{\cot \theta} = \frac{\operatorname{cosec}^2 \theta}{\cot \theta},$$

and

$$y = \operatorname{cosec} \theta - \frac{1}{\operatorname{cosec} \theta} = \frac{\operatorname{cosec}^2 \theta - 1}{\operatorname{cosec} \theta} = \frac{\cot^2 \theta}{\operatorname{cosec} \theta};$$

$$\therefore x^2 y = \operatorname{cosec}^3 \theta, \text{ and } xy^2 = \cot^3 \theta.$$

But

$$\operatorname{cosec}^2 \theta - \cot^2 \theta = 1;$$

$$\therefore x^3 y^3 - x^2 y^4 = 1.$$

## ELIMINATION.

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7. Here

$$a^3 = \frac{1}{\sin \theta} - \sin \theta = \frac{\cos^2 \theta}{\sin \theta},$$

and

$$b^3 = \frac{1}{\cos \theta} - \cos \theta = \frac{\sin^2 \theta}{\cos \theta};$$

and

$$\therefore a^6 b^3 = \cos^3 \theta, \text{ or } a^2 b = \cos \theta;$$

$$a^3 b^6 = \sin^3 \theta, \text{ or } ab^2 = \sin \theta;$$

$$\therefore a^4 b^2 + a^2 b^4 = 1.$$

8. By substituting for  $\cos 3\theta$  and  $\sin 3\theta$ , we have

$$4x = 4a \cos^3 \theta, \text{ or } x^{\frac{1}{3}} = a^{\frac{1}{3}} \cos \theta,$$

and

$$4y = 4a \sin^3 \theta, \text{ or } y^{\frac{1}{3}} = a^{\frac{1}{3}} \sin \theta;$$

$$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

9. By transposition, we have

$$x(1 + \tan^2 \theta) = a \tan^3 \theta, \text{ or } x = \frac{a \tan^3 \theta}{\sec^2 \theta},$$

and

$$y(\sec^2 \theta - 1) = a \sec^3 \theta, \text{ or } y = \frac{a \sec^3 \theta}{\tan^2 \theta};$$

$$\therefore x^3 y^2 = a^5 \tan^5 \theta, \text{ and } x^2 y^3 = a^5 \sec^5 \theta.$$

$$\therefore (x^2 y^3)^{\frac{2}{5}} - (x^3 y^2)^{\frac{2}{5}} = a^2 (\sec^2 \theta - \tan^2 \theta) = a^2.$$

10. Here

$$x = a \cos \theta (2 \cos 2\theta - 1) = a \cos \theta (4 \cos^2 \theta - 3) \\ = a \cos 3\theta.$$

Similarly

$$y = b \sin 3\theta.$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

11. Here

and

$$\sin \theta \sin \alpha + \cos \theta \cos \alpha = a,$$

$$\sin \theta \cos \beta - \cos \theta \sin \beta = b;$$

$$\therefore \sin \theta \cos (\alpha - \beta) = a \sin \beta + b \cos \alpha,$$

$$\cos \theta \cos (\alpha - \beta) = a \cos \beta - b \sin \alpha;$$

and

$$\therefore \cos^2 (\alpha - \beta) = a^2 + b^2 - 2ab \sin (\alpha - \beta).$$

12. Here

and

$$x + y = 3 - (1 - 2 \sin^2 2\theta) = 2 + 2 \sin^2 2\theta,$$

$$x - y = 4 \sin 2\theta;$$

$$\therefore x = 1 + 2 \sin 2\theta + \sin^2 2\theta = (1 + \sin 2\theta)^2,$$

$$x^{\frac{1}{2}} = 1 + \sin 2\theta.$$

or

$$y^{\frac{1}{2}} = 1 - \sin 2\theta.$$

Similarly,

$$\therefore x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2.$$

13. By addition and subtraction, we have

$$\begin{aligned}x + y &= (\sin \theta + \cos \theta) (1 + \sin 2\theta) \\&= (\sin \theta + \cos \theta)^3,\end{aligned}$$

and

$$\begin{aligned}x - y &= (\sin \theta - \cos \theta) (1 - \sin 2\theta) \\&= (\sin \theta - \cos \theta)^3;\end{aligned}$$

$$\begin{aligned}\therefore (x + y)^2 + (x - y)^2 &= (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 \\&= 2.\end{aligned}$$

14. Here

$$a^2 = 1 + \sin 2\theta,$$

$$\therefore a^2 - 1 + \cos 2\theta = b.$$

$$\therefore \sin 2\theta = a^2 - 1, \text{ and } \cos 2\theta = -(a^2 - b - 1);$$

$$\therefore (a^2 - 1)^2 + (a^2 - b - 1)^2 = 1.$$

15. Here

$$\begin{aligned}a &= (4 \cos^3 \theta - 3 \cos \theta) + (3 \sin \theta - 4 \sin^3 \theta) \\&= (\cos \theta - \sin \theta) \{4 (\cos^2 \theta + \cos \theta \sin \theta + \sin^2 \theta) - 3\} \\&= b (1 + 2 \sin 2\theta).\end{aligned}$$

But

$$1 - \sin 2\theta = b^2;$$

$$\therefore a = b (1 + 2 - 2b^2) = 3b - 2b^3.$$

16. By squaring the first equation, and multiplying by 2, we have

$$a^2 (1 + \cos 2\theta) + b^2 (1 - \cos 2\theta) - 2ab \sin 2\theta = 2c^2;$$

$$\therefore (a^2 - b^2) \cos 2\theta - 2ab \sin 2\theta = 2c^2 - a^2 - b^2.$$

From the second equation,

$$(a^2 - b^2) \sin 2\theta + 2ab \cos 2\theta = 2c^2.$$

By squaring and adding, we obtain

$$(a^2 - b^2)^2 + 4a^2b^2 = 4c^4 - 4c^2 (a^2 + b^2) + (a^2 + b^2)^2 + 4c^4;$$

$$\therefore 0 = 8c^4 - 4c^2 (a^2 + b^2);$$

$$\therefore a^2 + b^2 = 2c^2.$$

17. By squaring and adding, we have

$$\begin{aligned}x^2 + y^2 &= a^2 + b^2 + 2ab (\cos \theta \cos 2\theta + \sin \theta \sin 2\theta) \\&= a^2 + b^2 + 2ab \cos \theta.\end{aligned}$$

Again,

$$x + b = a \cos \theta + 2b \cos^2 \theta = \cos \theta (a + 2b \cos \theta),$$

and

$$y = \sin \theta (a + 2b \cos \theta);$$

$$\therefore (x + b)^2 + y^2 = (a + 2b \cos \theta)^2$$

$$= \frac{1}{a^2} (x^2 + y^2 - b^2)^2.$$

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18. Componendo and dividendo, we have

$$\frac{a}{b} = \frac{\tan(\theta + \alpha) + \tan(\theta - \alpha)}{\tan(\theta + \alpha) - \tan(\theta - \alpha)} = \frac{\sin 2\theta}{\sin 2\alpha};$$

$$\therefore b \sin 2\theta = a \sin 2\alpha.$$

Also

$$b \cos 2\theta = c - a \cos 2\alpha;$$

$$\therefore b^2 = c^2 - 2ac \cos 2\alpha + a^2.$$

19. By squaring and adding, we have

$$\begin{aligned} x^2 + y^2 &= a^2 \{2 - 2(\cos 3\theta \cos \theta + \sin 3\theta \sin \theta)\} \\ &= a^2 (2 - 2 \cos 2\theta) \\ &= 4a^2 \sin^2 \theta. \end{aligned}$$

And

$$2a^2 - x^2 - y^2 = 2a^2 \cos 2\theta.$$

But

$$x = 2a \cos 2\theta \sin \theta;$$

$$\begin{aligned} \therefore 4a^4 x^2 &= (2a^2 \cos 2\theta)^2 (4a^2 \sin^2 \theta) \\ &= (2a^2 - x^2 - y^2)^2 (x^2 + y^2). \end{aligned}$$

20. Solving the given equations for  $x$  and  $y$ , we have

$$\begin{aligned} x &= a (\cos \theta \cos 2\theta + 2 \sin \theta \sin 2\theta); \\ \therefore 2x &= a \{(\cos 3\theta + \cos \theta) + 2(\cos \theta - \cos 3\theta)\} \\ &= a (3 \cos \theta - \cos 3\theta) = a (6 \cos \theta - 4 \cos^3 \theta). \end{aligned}$$

And

$$\begin{aligned} y &= a (2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta); \\ \therefore 2y &= a \{2(\sin 3\theta + \sin \theta) - (\sin 3\theta - \sin \theta)\} \\ &= a (3 \sin \theta + \sin 3\theta) = a (6 \sin \theta - 4 \sin^3 \theta). \\ \therefore 2(x + y) &= a (\cos \theta + \sin \theta) \{6 - 4(\cos^2 \theta - \cos \theta \sin \theta + \sin^2 \theta)\} \\ &= 2a (\cos \theta + \sin \theta) (1 + \sin 2\theta); \\ \therefore x + y &= a (\cos \theta + \sin \theta)^3. \end{aligned}$$

Similarly,

$$\begin{aligned} x - y &= a (\cos \theta - \sin \theta)^3; \\ \therefore (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} &= 2a^{\frac{2}{3}}. \end{aligned}$$

21. From the first equation,

$$(x \sin \theta - y \cos \theta)^2 = (x^2 + y^2) (\sin^2 \theta + \cos^2 \theta);$$

whence

$$x \cos \theta + y \sin \theta = 0;$$

$$\therefore \frac{\cos \theta}{y} = \frac{\sin \theta}{-x} = \frac{1}{\sqrt{x^2 + y^2}}.$$

By substituting in the second equation, we have

$$\frac{1}{x^2 + y^2} \left( \frac{y^2}{a^2} + \frac{x^2}{b^2} \right) = \frac{1}{x^2 + y^2};$$

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

22. From the first equation, we have

$$bx \cos \theta + ay \sin \theta = ab \sqrt{\sin^2 \theta + \cos^2 \theta};$$

$$\therefore b^2 (x^2 - a^2) \cos^2 \theta + 2abxy \cos \theta \sin \theta + a^2 (y^2 - b^2) \sin^2 \theta = 0.$$

From the second equation, we have

$$(y^2 - b^2) \cos^2 \theta - 2xy \cos \theta \sin \theta + (x^2 - a^2) \sin^2 \theta = 0.$$

Multiplying this equation by  $ab$  and adding to the preceding equation,

$$\{b^2 (x^2 - a^2) + ab (y^2 - b^2)\} \cos^2 \theta + \{a^2 (y^2 - b^2) + ab (x^2 - a^2)\} \sin^2 \theta = 0;$$

$$\therefore \{bx^2 + ay^2 - ab(a + b)\} (b \cos^2 \theta + a \sin^2 \theta) = 0.$$

$$\therefore bx^2 + ay^2 - ab(a + b) = 0,$$

or

$$b \cos^2 \theta + a \sin^2 \theta = 0.$$

From the last result,

$$\frac{\cos^2 \theta}{a} = \frac{\sin^2 \theta}{-b};$$

but

$$\{(y^2 - b^2) \cos^2 \theta + (x^2 - a^2) \sin^2 \theta\}^2 = 4x^2 y^2 \cos^2 \theta \sin^2 \theta;$$

$$\therefore \{a(y^2 - b^2) - b(x^2 - a^2)\}^2 = -4abx^2 y^2.$$

23. We have  $4 \cos (\alpha - 3\theta) = m (3 \cos \theta + \cos 3\theta),$

and  $4 \sin (\alpha - 3\theta) = m (3 \sin \theta - \sin 3\theta);$

whence by squaring and adding, we have

$$16 = m^2 (10 + 6 \cos 4\theta).$$

Again, by multiplying the first equation by  $\cos 3\theta$  and the second by  $\sin 3\theta$  and subtracting, we obtain

$$4 \cos \alpha = m (3 \cos 4\theta + 1);$$

$$\therefore 16 = 10m^2 + 2m (4 \cos \alpha - m),$$

or

$$2 = m^2 + m \cos \alpha.$$

24. We have  $\frac{x}{y} = \frac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi,$

But  $\tan \alpha = \tan (\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi};$

$$\therefore \tan \alpha = \frac{xy}{y - x}.$$

25. We have  $a^2 + b^2 = 2 + 2 \cos (\theta - \phi);$

$$\therefore a^2 + b^2 = 2 + 2 \cos \alpha.$$

26. We have  $a \sin^2 \theta + b \cos^2 \theta = 1 = \sin^2 \theta + \cos^2 \theta;$

$$\therefore (a - 1) \tan^2 \theta = 1 - b.$$

Again,  $a \cos^2 \phi + b \sin^2 \phi = 1 = \cos^2 \phi + \sin^2 \phi$ ;  
 $\therefore (b-1) \tan^2 \phi = 1-a$ .

But  $a^2 \tan^2 \theta = b^2 \tan^2 \phi$ ;  
 $\therefore \frac{a^2(1-b)}{a-1} = \frac{b^2(1-a)}{b-1}$ ;  
 $\therefore a(b-1) = \pm b(a-1)$ .

Rejecting the upper sign, we have  $a+b=2ab$ .

27. From the first two equations, we have

$$\frac{\frac{x}{a}}{\cos \frac{\theta+\phi}{2}} = \frac{\frac{y}{b}}{\sin \frac{\theta+\phi}{2}} = \frac{1}{\cos \frac{\theta-\phi}{2}};$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \frac{\theta-\phi}{2} = \sec^2 \frac{\alpha}{2}.$$

28. We have  $\frac{a}{b} = \frac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi$ .

But  $\tan \alpha = \tan (\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$ ;

$$\therefore (a+b) \tan \alpha = b (\tan \theta - \tan \phi);$$

$$\therefore (a+b)^2 \tan^2 \alpha = b^2 \{ (\tan \theta + \tan \phi)^2 - 4 \tan \theta \tan \phi \}$$

$$= b^2 a^2 - 4ab.$$

29. We have  $\frac{a \cos^2 \theta + b \sin^2 \theta}{a \sin^2 \theta + b \cos^2 \theta} = \frac{m \cos^2 \phi}{n \sin^2 \phi}$ ;

$$\therefore \frac{a + b \tan^2 \theta}{a \tan^2 \theta + b} = \frac{m}{n \tan^2 \phi} = \frac{1}{\tan^2 \theta};$$

$$\therefore b \tan^4 \theta = b, \text{ or } \tan^2 \theta = \pm 1.$$

$$\therefore n \tan^2 \phi = \pm m.$$

By adding together the first two equations, we obtain

$$a+b = m \cos^2 \phi + n \sin^2 \phi.$$

If  $n \tan^2 \phi = m$ , then  $\frac{\cos^2 \phi}{n} = \frac{\sin^2 \phi}{m} = \frac{1}{m+n}$ ;

$$\therefore a+b = \frac{2mn}{m+n}.$$

If  $n \tan^2 \phi = -m$ , we obtain  $a+b=0$ .

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30. From the second and third equations, we have by addition

$$2 \cos \phi (x \cos \theta + y \sin \theta) = 6a;$$

$$\therefore 2a \sqrt{3} \cos \phi = 3a, \text{ whence } \cos \phi = \frac{\sqrt{3}}{2}.$$

From the second and third equations, we have by subtraction

$$2 \sin \phi (-x \sin \theta + y \cos \theta) = 2a;$$

$$\therefore -x \sin \theta + y \cos \theta = \pm 2a.$$

But

$$x \cos \theta + y \sin \theta = 2a \sqrt{3};$$

$$\therefore x^2 + y^2 = 16a^2.$$

31. We have  $c \sin \theta = a (\sin \theta \cos \phi + \cos \theta \sin \phi);$

$$\therefore (c - a \cos \phi) \sin \theta = a \sin \phi \cos \theta = b \sin \theta \cdot \cos \theta;$$

$$\therefore c - a \cos \phi = b \cos \theta = b (2m + \cos \phi);$$

$$\therefore \cos \phi = \frac{c - 2bm}{a + b}.$$

And

$$\cos \theta = 2m + \cos \phi = \frac{c + 2am}{a + b}.$$

But

$$a \sin \phi = b \sin \theta;$$

$$\therefore a^2 - a^2 \cos^2 \phi = b^2 - b^2 \cos^2 \theta;$$

$$\therefore a^2 (a + b)^2 - a^2 (c - 2bm)^2 = b^2 (a + b)^2 - b^2 (c + 2am)^2;$$

$$\therefore (a^2 - b^2) (a + b)^2 = c^2 (a^2 - b^2) - 4abcm (a + b);$$

$$\therefore 4abcm = (a - b) \{c^2 - (a + b)^2\}.$$

### EXAMPLES. XXV. c. PAGE 334.

1. By putting  $x = y \cos \theta$ , the given equation becomes

$$\cos^3 \theta - \frac{3}{y^2} \cos \theta - \frac{1}{y^3} = 0.$$

But

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore \frac{3}{y^2} = \frac{3}{4}; \text{ whence } y = 2.$$

Also

$$\frac{\cos 3\theta}{4} = \frac{1}{y^3} = \frac{1}{8}; \text{ whence } \cos 3\theta = \frac{1}{2};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 60^\circ;$$

$$\therefore \theta = 20^\circ, 100^\circ, \text{ or } 140^\circ.$$

But  $x = y \cos \theta = 2 \cos \theta$ ; and therefore the roots are

$$2 \cos 20^\circ, \quad -2 \cos 40^\circ, \quad -2 \cos 80^\circ.$$

2. By putting  $x = y \sin \theta$ , the given equation becomes

$$\sin^3 \theta - \frac{3}{y^2} \sin \theta + \frac{1}{y^3} = 0.$$

But

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$$

$$\therefore y^2 = 4; \text{ whence } y = 2.$$

Also

$$\frac{\sin 3\theta}{4} = \frac{1}{y^3} = \frac{1}{8}; \text{ whence } \sin 3\theta = \frac{1}{2}.$$

Hence as in Art. 328, the roots of the equation are

$$2 \sin 10^\circ, \quad 2 \sin 50^\circ, \quad -2 \sin 70^\circ.$$

3. Put  $x = y \cos \theta$ , then  $\cos^3 \theta - \frac{3}{y^2} \cos \theta - \frac{\sqrt[3]{3}}{y^3} = 0.$

But

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore y^2 = 4; \text{ whence } y = 2.$$

Also

$$\frac{\cos 3\theta}{4} = \frac{\sqrt[3]{3}}{y^3} = \frac{\sqrt[3]{3}}{8}; \text{ whence } \cos 3\theta = \frac{\sqrt[3]{3}}{2};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 30^\circ; \quad \therefore \theta = 10^\circ, 110^\circ, \text{ or } 130^\circ.$$

But  $x = y \cos \theta = 2 \cos \theta$ , and therefore the roots are

$$2 \cos 10^\circ, \quad -2 \cos 50^\circ, \quad -2 \cos 70^\circ.$$

4. Put  $x = y \sin \theta$ ; then  $\sin^3 \theta - \frac{3}{4y^2} \sin \theta + \frac{\sqrt[3]{2}}{8y^3} = 0.$

But

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$$

$$\therefore 4y^2 = 4; \text{ whence } y = 1.$$

Also since  $\frac{\sin 3\theta}{4} = \frac{\sqrt[3]{2}}{8y^3} = \frac{\sqrt[3]{2}}{8}; \text{ whence } \sin 3\theta = \frac{1}{\sqrt[3]{2}};$

$$\therefore 3\theta = n \cdot 180^\circ + (-1)^n 45^\circ;$$

$$\therefore \theta = 15^\circ, 45^\circ, 135^\circ, 165^\circ, 255^\circ, \dots$$

$$\therefore \sin \theta = \sin 15^\circ, \sin 45^\circ, \sin 255^\circ.$$

But  $x = y \sin \theta = \sin \theta$ , and therefore the roots are

$$\sin 15^\circ, \quad \sin 45^\circ, \quad -\sin 75^\circ.$$

5. Put  $x = y \sin \theta$ ; then  $\sin^3 \theta - \frac{3}{4a^2y^2} \sin \theta + \frac{\sin 3\theta}{4a^3y^3} = 0.$

But

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$$

$$\therefore 4a^2y^2 = 4; \text{ whence } y = \frac{1}{a}.$$

Also 
$$\frac{\sin 3\theta}{4} = \frac{\sin 3A}{4a^3y^3} = \frac{\sin 3A}{4};$$

$$\therefore 3\theta = n \cdot 180^\circ + (-1)^n 3A;$$

$$\therefore \theta = A, \quad 60^\circ - A, \quad 120^\circ + A, \quad 180^\circ - A, \quad 240^\circ + A, \dots$$

$$\therefore \sin \theta = \sin A, \quad \sin (60^\circ - A), \quad \sin (240^\circ + A).$$

But  $x = y \sin \theta = \frac{1}{a} \sin \theta$ , and therefore the roots are

$$\frac{1}{a} \sin A, \quad \frac{1}{a} \sin (60^\circ - A), \quad -\frac{1}{a} \sin (60^\circ + A).$$

6. Put  $x = y \cos \theta$ ; then  $\cos^3 \theta - \frac{3a^2}{y^2} \cos \theta - \frac{2a^3 \cos 3A}{y^3} = 0$ .

But 
$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore \frac{3a^2}{y^2} = \frac{3}{4}; \text{ whence } y = 2a.$$

Also 
$$\frac{\cos 3\theta}{4} = \frac{2a^3 \cos 3A}{y^3} = \frac{\cos 3A}{4};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 3A;$$

$$\therefore \theta = A, \quad 120^\circ \pm A.$$

But  $x = y \cos \theta = 2a \cos \theta$ , and therefore the roots are

$$2a \cos A, \quad 2a \cos (120^\circ \pm A).$$

7. (1) From the theory of quadratic equations, we have

$$\sin \alpha + \sin \beta = -\frac{b}{a} \dots \dots \dots (1).$$

By supposition,  $\sin \alpha + 2 \sin \beta = 1,$

$$\therefore \sin \beta = 1 + \frac{b}{a}.$$

But  $a \sin^2 \beta + b \sin \beta + c = 0 \dots \dots \dots (2),$

$$\therefore (a+b)^2 + b(a+b) + ac = 0.$$

(2) Substituting from the equation  $c \sin \alpha = a \sin \beta$  in (1), we have

$$a \sin \beta + c \sin \beta = -\frac{bc}{a};$$

$$\therefore a(a+c) \sin \beta = -bc.$$

But  $\sin \alpha \sin \beta = \frac{c}{a};$

$$\therefore a^2 \sin^2 \beta = c^2; \text{ whence } a+c = \pm b.$$

8. We have  $\tan \alpha + \tan \beta = \frac{b}{a}$ .

Also, by hypothesis,  $a \tan \alpha + b \tan \beta = 2b$ ;

whence  $(b-a) \tan \beta = b$ .

But  $a \tan^2 \beta - b \tan \beta + c = 0$ ;

$$\therefore ab^2 - b^2(b-a) + c(b-a)^2 = 0.$$

9. We have  $\tan \alpha + \tan \beta + \tan \gamma = 0$ ,

and  $\tan \alpha \tan \beta + \tan \alpha \tan \gamma + \tan \beta \tan \gamma = \frac{2a-x}{a}$ ;

$$\therefore \tan \alpha \tan \beta - \tan^2 \gamma = \frac{2a-x}{a}.$$

Now  $\tan^2 \alpha + \tan^2 \beta = (\tan \alpha + \tan \beta)^2 - 2 \tan \alpha \tan \beta$   
 $= \tan^2 \gamma - 2 \tan^2 \gamma - \frac{2(2a-x)}{a}$ ;

$$\therefore a(\tan^2 \alpha + \tan^2 \beta) = -a \tan^2 \gamma - 4a + 2x.$$

But, by hypothesis,  $a(\tan^2 \alpha + \tan^2 \beta) = 2x - 5a$ ;

$$\therefore 2x - 5a = -a \tan^2 \gamma - 4a + 2x;$$

$$\therefore \tan^2 \gamma = 1.$$

But  $a \tan^3 \gamma + (2a-x) \tan \gamma + y = 0$ ;

$$\therefore a \tan \gamma + (2a-x) \tan \gamma + y = 0;$$

$$\therefore (3a-x) \tan \gamma + y = 0;$$

$$\therefore 3a-x = \pm y.$$

10. We have  $\cos \alpha \cos \beta + \cos \alpha \cos \gamma + \cos \beta \cos \gamma = b$ .

Also, by supposition,  $\cos \alpha \cos \beta + \cos \alpha \cos \gamma = 2b$ ;

$$\therefore \cos \beta \cos \gamma = -b.$$

Again,  $\cos \alpha \cos \beta \cos \gamma = -c$ ;

$$\therefore b \cos \alpha = c.$$

Now  $\cos^3 \alpha + a \cos^2 \alpha + b \cos \alpha + c = 0$ ;

$$\therefore c^3 + abc^2 + b^3c + b^3c = 0.$$

11. As on page 329 we may shew that

$$\cos \alpha, \quad \cos \left( \frac{2\pi}{3} + \alpha \right), \quad \cos \left( \frac{2\pi}{3} - \alpha \right)$$

are the roots of the cubic

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\alpha}{4} = 0.$$

Write  $\cos \theta = \frac{1}{\sec \theta}$ ; then  $\frac{\cos 3\alpha}{4} \sec^3 \theta + \frac{3}{4} \sec^2 \theta - 1 = 0$ ;

$$\therefore \sec \alpha + \sec \left( \frac{2\pi}{3} + \alpha \right) + \sec \left( \frac{2\pi}{3} - \alpha \right) = -\frac{3}{4} \div \frac{\cos 3\alpha}{4}.$$

12. The values of  $\sin \theta$  found from the equation  $\sin 3\theta = \sin 3\alpha$  are

$$\sin \alpha, \quad \sin \left( \frac{2\pi}{3} + \alpha \right), \quad \sin \left( \frac{4\pi}{3} + \alpha \right);$$

that is, these three quantities are the roots of the cubic equation

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\alpha}{4} = 0 \dots\dots\dots(1).$$

$\therefore S_1 = \text{Sum of the roots} = 0$ ;

and  $S_2 = \text{Sum of products of the roots taken two at a time} = -\frac{3}{4}$ ;

$$\begin{aligned} \therefore \sin^2 \alpha + \sin^2 \left( \frac{2\pi}{3} + \alpha \right) + \sin^2 \left( \frac{4\pi}{3} + \alpha \right) &= S_1^2 - 2S_2 \\ &= 0 - 2 \left( -\frac{3}{4} \right) = \frac{3}{2}. \end{aligned}$$

13. In equation (1) of Example 12, put  $\sin \theta = \frac{1}{\operatorname{cosec} \theta}$ ; then

$$\frac{\sin 3\alpha}{4} \operatorname{cosec}^3 \theta - \frac{3}{4} \operatorname{cosec}^2 \theta + 1 = 0.$$

$$\therefore \operatorname{cosec} \alpha + \operatorname{cosec} \left( \frac{2\pi}{3} + \alpha \right) + \operatorname{cosec} \left( \frac{4\pi}{3} + \alpha \right) = \frac{3}{4} \div \frac{\sin 3\alpha}{4}.$$

14. On p. 330 we have shewn that  $\sin^2 \frac{\pi}{5}$  and  $\sin^2 \frac{2\pi}{5}$  are the roots of the quadratic equation  $16x^2 - 20x + 5 = 0$ .

Put  $x = \frac{1}{y}$ ; then  $5y^2 - 20y + 16 = 0$ , is an equation whose roots are

$$\operatorname{cosec}^2 \frac{\pi}{5} \text{ and } \operatorname{cosec}^2 \frac{2\pi}{5};$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{5} + \operatorname{cosec}^2 \frac{2\pi}{5} = \frac{20}{5} = 4.$$

15. If

$$\cos 3\theta = \cos 2\theta,$$

then

$$4 \cos^3 \theta - 2 \cos^2 \theta - 3 \cos \theta + 1 = 0,$$

or

$$(\cos \theta - 1)(4 \cos^2 \theta + 2 \cos \theta - 1) = 0.$$

But the roots of  $\cos 3\theta = \cos 2\theta$  considered as a cubic in  $\cos \theta$  are

$$1, \quad \cos \frac{2\pi}{5}, \quad \cos \frac{4\pi}{5}. \quad [\text{Art. 331.}]$$

$\therefore \cos \frac{2\pi}{5}$  and  $\cos \frac{4\pi}{5}$  are the roots of  $4 \cos^2 \theta + 2 \cos \theta - 1 = 0$ ;

$$\therefore \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{2}{4} = -\frac{1}{2},$$

and

$$\cos \frac{2\pi}{5} \cos \frac{4\pi}{5} = -\frac{1}{4}.$$

16. (1) Let  $7\theta = n\pi$ , where  $n$  is any odd integer;

then

$$4\theta = n\pi - 3\theta, \text{ and } \cos 4\theta = -\cos 3\theta.$$

$$\therefore 8 \cos^4 \theta - 8 \cos^2 \theta + 1 = -4 \cos^3 \theta + 3 \cos \theta;$$

$$\therefore (\cos \theta + 1)(8 \cos^3 \theta - 4 \cos^2 \theta - 4 \cos \theta + 1) = 0. \quad (1).$$

The roots of the equation  $\cos 4\theta = -\cos 3\theta$  considered as a biquadratic in  $\cos \theta$  are  $\cos \frac{\pi}{7}$ ,  $\cos \frac{3\pi}{7}$ ,  $\cos \frac{5\pi}{7}$ ,  $\cos \frac{7\pi}{7}$ , the last of which corresponds to the factor  $\cos \theta + 1$  in equation (1). Hence the equation whose roots are  $\cos \frac{\pi}{7}$ ,  $\cos \frac{3\pi}{7}$ ,  $\cos \frac{5\pi}{7}$  is

$$8 \cos^3 \theta - 4 \cos^2 \theta - 4 \cos \theta + 1 = 0.$$

(2) Let  $y$  denote any one of the quantities

$$\sin^2 \frac{\pi}{14}, \quad \sin^2 \frac{3\pi}{14}, \quad \sin^2 \frac{5\pi}{14}.$$

then  $2y = 1 - x$ , where  $x$  denotes one of the quantities

$$\cos \frac{\pi}{7}, \quad \cos \frac{3\pi}{7}, \quad \cos \frac{5\pi}{7}.$$

But we have seen in the first part of this question that these quantities are the roots of the cubic

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Substituting  $x = 1 - 2y$ , we have

$$8(1 - 2y)^3 - 4(1 - 2y)^2 - 4(1 - 2y) + 1 = 0,$$

or

$$64y^3 - 80y^2 + 24y - 1 = 0.$$

17. Let  $y$  denote one of the quantities  $\sin^2 \frac{\pi}{7}$ ,  $\sin^2 \frac{2\pi}{7}$ ,  $\sin^2 \frac{3\pi}{7}$ , then  $2y = 1 - x$ ; where  $x$  denotes one of the quantities

$$\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}.$$

But in Art. 331, we have shewn that these quantities are the roots of the cubic

$$8x^3 + 4x^2 - 4x - 1 = 0.$$

Substituting  $x = 1 - 2y$ , we have

$$8(1 - 2y)^3 + 4(1 - 2y)^2 - 4(1 - 2y) - 1 = 0,$$

or

$$64y^3 - 112y^2 + 56y - 7 = 0,$$

the roots of which are  $\sin^2 \frac{\pi}{7}$ ,  $\sin^2 \frac{2\pi}{7}$ ,  $\sin^2 \frac{3\pi}{7}$ .

$$\therefore \sin^4 \frac{\pi}{7} + \sin^4 \frac{2\pi}{7} + \sin^4 \frac{3\pi}{7} = \left(\frac{112}{64}\right)^2 - 2\left(\frac{56}{64}\right) = \frac{21}{16}.$$

Put  $y = \frac{1}{z}$ ; then  $7z^3 - 56z^2 + 112z - 64 = 0$  is an equation whose roots are

$$\operatorname{cosec}^2 \frac{\pi}{7}, \operatorname{cosec}^2 \frac{2\pi}{7}, \operatorname{cosec}^2 \frac{3\pi}{7}.$$

$$\therefore \operatorname{cosec}^4 \frac{\pi}{7} + \operatorname{cosec}^4 \frac{2\pi}{7} + \operatorname{cosec}^4 \frac{3\pi}{7} = \left(\frac{56}{7}\right)^2 - 2\left(\frac{112}{7}\right) = 32.$$

18. (1) As in Art. 331 the required equation is  $\cos 5\theta = \cos 4\theta$ .

Expressing  $\cos 5\theta$  and  $\cos 4\theta$  in terms of  $\cos \theta$ , we have

$$16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1. \quad [\text{See Art. 332.}]$$

By transposition and removal of the factor  $\cos \theta - 1$  we obtain

$$16 \cos^4 \theta + 8 \cos^3 \theta - 12 \cos^2 \theta - 4 \cos \theta + 1 = 0,$$

which is the equation required.

(2) Put  $\cos \theta = x$ , then the above equation becomes

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

Now  $\cos \frac{\pi}{9} = -\cos \frac{8\pi}{9}$ ,  $\cos \frac{3\pi}{9} = -\cos \frac{6\pi}{9}$ , ...;

hence by writing  $x = -y$ , we see that

$$\cos \frac{\pi}{9}, \cos \frac{3\pi}{9}, \cos \frac{5\pi}{9}, \cos \frac{7\pi}{9}$$

are the roots of the equation  $16y^4 - 8y^3 - 12y^2 + 4y + 1 = 0$ .

The same result may be arrived at by putting  $9\theta = n\pi$ , where  $n$  is any odd integer. For  $5\theta = n\pi - 4\theta$ , so that  $\cos 5\theta = -\cos 4\theta$ . [See Example 16 (1).]

19. Let  $y$  denote one of the quantities

$$\cos^2 \frac{\pi}{9}, \cos^2 \frac{2\pi}{9}, \cos^2 \frac{3\pi}{9}, \cos^2 \frac{4\pi}{9} \dots \dots \dots (1),$$

then  $2y = 1 + x$ , where  $x$  denotes one of the quantities

$$\cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{6\pi}{9}, \cos \frac{8\pi}{9}.$$

But these quantities are by the last example the roots of the equation

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0;$$

hence the given quantities are the roots of

$$16(2y - 1)^4 + 8(2y - 1)^3 - 12(2y - 1)^2 - 4(2y - 1) + 1 = 0,$$

or

$$256y^4 - 448y^3 + 240y^2 - 40y + 1 = 0.$$

$$\begin{aligned} \therefore \cos^4 \frac{\pi}{9} + \cos^4 \frac{2\pi}{9} + \cos^4 \frac{3\pi}{9} + \cos^4 \frac{4\pi}{9} &= \left( \frac{448}{256} \right)^2 - 2 \left( \frac{240}{256} \right) \\ &= \frac{49}{16} - \frac{30}{16} = \frac{19}{16}. \end{aligned}$$

If we put  $y = \frac{1}{z}$ , we obtain

$$z^4 - 40z^3 + 240z^2 - 448z + 256 = 0,$$

the roots of which are

$$\sec^2 \frac{\pi}{9}, \sec^2 \frac{2\pi}{9}, \sec^2 \frac{3\pi}{9}, \sec^2 \frac{4\pi}{9}.$$

$$\therefore \sec^4 \frac{\pi}{9} + \sec^4 \frac{2\pi}{9} + \sec^4 \frac{3\pi}{9} + \sec^4 \frac{4\pi}{9} = (40)^2 - 2 \times 240 = 1120.$$

20. As on p. 332, we may shew that the given quantities are the roots of the equation  $\tan 5\theta = -\tan 4\theta$ , considered as an equation in  $\tan \theta$ .

Put  $\tan \theta = t$ , then  $\tan 5\theta = \tan (3\theta + 2\theta)$ ;

$$\therefore \tan 5\theta = \frac{\frac{3t - t^3}{1 - 3t^2} + \frac{2t}{1 - t^2}}{1 - \frac{2t(3t - t^3)}{(1 - 3t^2)(1 - t^2)}} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}.$$

Hence the equation  $\tan 5\theta = -\tan 4\theta$  becomes

$$\frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} = -\frac{4t^3 - 4t}{t^4 - 6t^2 + 1} = 0;$$

$$\therefore (t^8 - 16t^6 + 66t^4 - 40t^2 + 5) - (20t^6 - 60t^4 + 4t^2 - 4) = 0;$$

$$\therefore t^8 - 36t^6 + 126t^4 - 84t^2 + 9 = 0,$$

which is a biquadratic in  $t^2$  having for roots

$$\tan^2 \frac{\pi}{9}, \tan^2 \frac{2\pi}{9}, \tan^2 \frac{3\pi}{9}, \tan^2 \frac{4\pi}{9}.$$

Put  $t^2 = \frac{1}{x}$ , then

$$9x^4 - 84x^3 + 126x^2 - 36 + 1 = 0,$$

an equation whose roots are

$$\cot^2 \frac{\pi}{9}, \cot^2 \frac{2\pi}{9}, \cot^2 \frac{3\pi}{9}, \cot^2 \frac{4\pi}{9}.$$

$$\text{But } \cot^2 \frac{3\pi}{9} = \cot^2 \frac{\pi}{3} = \frac{1}{3}.$$

$$\therefore \cot^2 \frac{\pi}{9} + \cot^2 \frac{2\pi}{9} + \cot^2 \frac{4\pi}{9} + \frac{1}{3} = \frac{84}{9} = 9\frac{1}{3}.$$

$$\therefore \cot^2 \frac{\pi}{9} + \cot^2 \frac{2\pi}{9} + \cot^2 \frac{4\pi}{9} = 9.$$

21. (1) As in Ex. 1, p. 332, we can shew that  $t^6 - 21t^4 + 35t^2 - 7 = 0$  is an equation whose roots are

$$\tan^2 \frac{\pi}{7}, \tan^2 \frac{2\pi}{7}, \tan^2 \frac{3\pi}{7};$$

$\therefore$  writing  $\frac{1}{c}$  for  $t$ , we obtain

$$7c^6 - 35c^4 + 21c^2 - 1 = 0,$$

which is a cubic in  $c^2$  whose roots are

$$\cot^2 \frac{\pi}{7}, \cot^2 \frac{2\pi}{7}, \cot^2 \frac{3\pi}{7}.$$

$$\therefore \cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} = 5;$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7} = 8.$$

$$(2) \text{ Here } \cos \frac{\pi}{11} = -\cos \frac{10\pi}{11}, \cos \frac{3\pi}{11} = -\cos \frac{8\pi}{11}, \cos \frac{5\pi}{11} = -\cos \frac{6\pi}{11}.$$

$$\therefore \cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \cos \frac{4\pi}{11} \cos \frac{5\pi}{11}$$

$$= -\cos \frac{2\pi}{11} \cos \frac{4\pi}{11} \cos \frac{6\pi}{11} \cos \frac{8\pi}{11} \cos \frac{10\pi}{11}.$$

Now  $\cos \frac{2\pi}{11}, \cos \frac{4\pi}{11}, \dots, \cos \frac{10\pi}{11}$  are the roots of the equation

$$\cos 6\theta = \cos 5\theta.$$

[See Art. 331.]

Using the expressions for  $\cos 6\theta$  and  $\cos 5\theta$  given in Art. 332 and putting  $x$  for  $\cos \theta$ , we have

$$32x^6 - 48x^4 + 18x^2 - 1 = 16x^5 - 20x^3 + 5x,$$

or

$$32x^6 - 16x^5 - 48x^4 + 20x^3 + 18x^2 - 5x - 1 = 0.$$

Removing the factor  $x - 1$ , which corresponds to  $\cos \theta = 1$ , we have

$$32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 = 0.$$

The product of the roots is  $-\frac{1}{32}$ , and therefore the value of the required expression is  $\frac{1}{32}$ .

## MISCELLANEOUS EXAMPLES. I. PAGE 336.

1. By transposition, we have

$$a \left( \tan a - \tan \frac{a+\beta}{2} \right) = b \left( \tan \frac{a+\beta}{2} - \tan \beta \right);$$

$$\therefore \frac{a \sin \left( a - \frac{a+\beta}{2} \right)}{\cos a \cos \frac{a+\beta}{2}} = \frac{b \sin \left( \frac{a+\beta}{2} - \beta \right)}{\cos \beta \cos \frac{a+\beta}{2}};$$

$$\therefore \frac{a}{\cos a} = \frac{b}{\cos \beta}.$$

2. We have

$$\frac{a+b}{a} \sin^4 a + \frac{a+b}{b} \cos^4 a = 1;$$

$$\therefore \left( 1 + \frac{b}{a} \right) \sin^4 a + \left( 1 + \frac{a}{b} \right) \cos^4 a = \sin^4 a + 2 \sin^2 a \cos^2 a + \cos^4 a;$$

$$\therefore \frac{b}{a} \sin^4 a - 2 \sin^2 a \cos^2 a + \frac{a}{b} \cos^4 a = 0;$$

$$\therefore \frac{\sin^4 a}{a^2} - \frac{2 \sin^2 a \cos^2 a}{ab} + \frac{\cos^4 a}{b^2} = 0;$$

$$\therefore \frac{\sin^2 a}{a} = \frac{\cos^2 a}{b} = \frac{1}{a+b}.$$

$$\therefore \frac{\sin^8 a}{a^3} + \frac{\cos^8 a}{b^3} = \frac{a}{(a+b)^4} + \frac{b}{(a+b)^4} = \frac{1}{(a+b)^3}.$$

$$\begin{aligned}
3. \quad \text{The left side} &= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \tan \left( \frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \left( \frac{\pi}{4} - \frac{\beta}{2} \right)} \\
&= \tan^{-1} \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \cos \left( \frac{\pi}{4} - \frac{\beta}{2} \right)}{\cos^2 \frac{\alpha}{2} \cos^2 \left( \frac{\pi}{4} - \frac{\beta}{2} \right) - \sin^2 \frac{\alpha}{2} \sin^2 \left( \frac{\pi}{4} - \frac{\beta}{2} \right)} \\
&= \tan^{-1} \frac{\frac{1}{2} \sin \alpha \sin \left( \frac{\pi}{2} - \beta \right)}{\cos \left\{ \frac{\alpha}{2} + \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \right\} \cos \left\{ \frac{\alpha}{2} - \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \right\}} \\
&\quad \text{[XI. a. Ex. 12.]} \\
&= \tan^{-1} \frac{\sin \alpha \cos \beta}{\cos \alpha + \cos \left( \frac{\pi}{2} - \beta \right)}.
\end{aligned}$$

4. By putting  $n=1$ , we have

$$\operatorname{cosec}^2 \alpha \sin^4 \theta + \sec^2 \alpha \cos^4 \theta = 1;$$

$$\therefore (1 + \cot^2 \alpha) \sin^4 \theta + (1 + \tan^2 \alpha) \cos^4 \theta = \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta;$$

$$\therefore \cot^2 \alpha \sin^4 \theta - 2 \sin^2 \theta \cos^2 \theta + \tan^2 \alpha \cos^4 \theta = 0;$$

$$\therefore \cot \alpha \sin^2 \theta - \tan \alpha \cos^2 \theta = 0;$$

$$\therefore \frac{\sin^2 \theta}{\sin^2 \alpha} = \frac{\cos^2 \theta}{\cos^2 \alpha} = 1;$$

$$\therefore \frac{\sin^{2n} \theta}{\sin^{2n} \alpha} = \frac{\cos^{2n} \theta}{\cos^{2n} \alpha} = 1;$$

$$\therefore \frac{\sin^{2n+2} \theta}{\sin^{2n} \alpha} + \frac{\cos^{2n+2} \theta}{\cos^{2n} \alpha} = \sin^2 \theta + \cos^2 \theta = 1.$$

5. We have  $(a \cos \theta + b \sin \theta)^2 = c^2 = c^2 (\cos^2 \theta + \sin^2 \theta);$

$$\therefore (a^2 - c^2) \cos^2 \theta + 2ab \cos \theta \sin \theta + (b^2 - c^2) \sin^2 \theta = 0.$$

Again,

$$a \cos^2 \theta + b \sin^2 \theta = c (\cos^2 \theta + \sin^2 \theta);$$

$$\therefore (a - c) \cos^2 \theta + (b - c) \sin^2 \theta = 0.$$

Hence by cross multiplication,

$$\frac{\cos^2 \theta}{2ab(b-c)} = \frac{\cos \theta \sin \theta}{(b^2 - c^2)(a-c) - (a^2 - c^2)(b-c)} = \frac{\sin^2 \theta}{-2ab(a-c)};$$

$$\therefore \frac{\cos^2 \theta}{2ab(b-c)} = \frac{\cos \theta \sin \theta}{(b-c)(a-c)(b-a)} = \frac{\sin^2 \theta}{-2ab(a-c)};$$

$$\therefore -4a^2b^2(b-c)(a-c) = (b-c)^2(a-c)^2(b-a)^2,$$

or

$$4a^2b^2 + (b-c)(a-c)(a-b)^2 = 0.$$

$$\begin{aligned}
 6. (i) \quad & 4\Sigma \sin(\beta - \gamma) \cos(\alpha - \beta) \cos(\alpha - \gamma) \\
 &= 2\Sigma \sin(\beta - \gamma) \{\cos(2\alpha - \beta - \gamma) + \cos(\beta - \gamma)\} \\
 &= 2\Sigma \sin(\beta - \gamma) \cos\{2\alpha - (\beta + \gamma)\} + 2\Sigma \sin(\beta - \gamma) \cos(\beta - \gamma) \\
 &= \Sigma \{\sin(2\alpha - 2\gamma) + \sin(2\beta - 2\alpha)\} + \Sigma \sin 2(\beta - \gamma) \\
 &= -2\Sigma \sin 2(\beta - \gamma) + \Sigma \sin 2(\beta - \gamma) \\
 &= -\Sigma \sin 2(\beta - \gamma) \\
 &= 4\Pi \sin(\beta - \gamma). \quad [\text{Ex. 2, page 304.}]
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad & 4\Sigma \sin \alpha \sin(\beta - \gamma) \cos(\beta + \gamma - \alpha) \\
 &= 2\Sigma \sin \alpha \{\sin(2\beta - \alpha) + \sin(\alpha - 2\gamma)\} \\
 &= 2\Sigma \sin \alpha \sin(2\beta - \alpha) + 2\Sigma \sin \alpha \sin(\alpha - 2\gamma) \\
 &= \Sigma \{\cos 2(\alpha - \beta) - \cos 2\beta\} + \Sigma \{\cos 2\gamma - \cos 2(\alpha - \gamma)\} \\
 &= \Sigma \{\cos 2(\alpha - \beta) - \cos 2(\gamma - \alpha)\} + \Sigma (\cos 2\gamma - \cos 2\beta) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad & 4\Sigma \sin \alpha \sin(\beta - \gamma) \sin(\beta + \gamma - \alpha) \\
 &= 2\Sigma \sin \alpha \{\cos(\alpha - 2\gamma) - \cos(2\beta - \alpha)\} \\
 &= \Sigma \{\sin(2\alpha - 2\gamma) + \sin 2\gamma\} - \Sigma \{\sin 2\beta + \sin(2\alpha - 2\beta)\} \\
 &= -\Sigma \sin 2(\gamma - \alpha) + \Sigma (\sin 2\gamma - \sin 2\beta) - \Sigma \sin 2(\alpha - \beta) \\
 &= -2\Sigma \sin 2(\alpha - \beta) \\
 &= 8\Pi \sin(\alpha - \beta). \quad [\text{Ex. 2, page 304.}]
 \end{aligned}$$

7. (1) Let  $x, y, z$  denote the lengths of  $PA, PB, PC$  respectively; and let the areas of the triangles  $PBC, PCA, PAB$  be denoted by  $\delta_1, \delta_2, \delta_3$  respectively.

Then in the triangle  $PBC$

$$\begin{aligned}
 \cot \omega &= \frac{\cos \omega}{\sin \omega} = \frac{a^2 + y^2 - z^2}{2ay \sin \omega} = \frac{a^2 + y^2 - z^2}{4\delta_1}; \\
 \therefore \cot \omega &= \frac{a^2 + y^2 - z^2}{4\delta_1} = \frac{b^2 + z^2 - x^2}{4\delta_2} = \frac{c^2 + x^2 - y^2}{4\delta_3} \\
 &= \frac{a^2 + b^2 + c^2}{4(\delta_1 + \delta_2 + \delta_3)} = \frac{a^2 + b^2 + c^2}{4\Delta} \\
 &= \cot A + \cot B + \cot C. \quad [\text{XVIII. a., Ex. 3.}]
 \end{aligned}$$

(2) By squaring the result just obtained, we have

$$\begin{aligned}
 \cot^2 \omega &= \cot^2 A + \cot^2 B + \cot^2 C + 2\Sigma \cot B \cot C \\
 &= \cot^2 A + \cot^2 B + \cot^2 C + 2,
 \end{aligned}$$

since

$$\Sigma \cot B \cot C = 1.$$

$$\therefore 1 + \cot^2 \omega = (1 + \cot^2 A) + (1 + \cot^2 B) + (1 + \cot^2 C);$$

$$\therefore \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

8. On page 195 of the *Elementary Trigonometry*, suppose that  $ABFD$  is a vertical plane running N. and S., and that  $CG$  is drawn in a S.E. direction. Also suppose that  $AB = a$ , and  $AC = 169a$ ;

then

$$BC^2 = (169a)^2 - a^2 = 28560a^2;$$

$$\therefore CG^2 = 2BC^2 = 57120a^2;$$

$$\therefore CH^2 = 57121a^2, \bullet$$

$$\therefore CH = 239a.$$

9. Let  $ABC$  be a horizontal section of the two walls, and let  $\phi$  be the inclination of the wall  $AB$  to the meridian, so that  $\gamma - \phi$  is the inclination of  $BC$  to the meridian.

The length of the shadow of the wall  $AB$  measured along the meridian is  $a \cot \theta$ ; hence the breadth of the shadow (which is measured at right angles to  $AB$ ) is  $a \cot \theta \sin \phi$ ,

$$\therefore b = a \cot \theta \sin \phi.$$

Similarly,

$$c = a \cot \theta \sin (\gamma - \phi);$$

$$\therefore c = a \cot \theta (\sin \gamma \cos \phi - \cos \gamma \sin \phi)$$

$$= a \cot \theta \sin \gamma \cos \phi - b \cos \gamma;$$

$$\therefore c + b \cos \gamma = a \cot \theta \sin \gamma \cos \phi.$$

Also

$$b \sin \gamma = a \cot \theta \sin \gamma \sin \phi.$$

By squaring and adding, we have

$$c^2 + 2bc \cos \gamma + b^2 = a^2 \cot^2 \theta \sin^2 \gamma.$$

## MISCELLANEOUS EXAMPLES. K.

1. If  $x$  is the number of degrees in the vertical angle  $x + 12x + 12x = 180$ , whence  $x = \frac{180}{25} = 7.2$ . Thus the angle is  $7^\circ 12'$ .

Again, the number of grades  $= \frac{200}{180} \times \frac{180}{25} = 8$ .

2. We have  $\frac{\alpha}{4} = \frac{\beta}{5} = \frac{\gamma}{6} = \frac{\alpha + \beta + \gamma}{15} = \frac{\pi}{15},$

whence  $\alpha = \frac{4\pi}{15}, \quad \beta = \frac{\pi}{3}, \quad \gamma = \frac{2\pi}{5}.$

3. The first side  $= \left( \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} \right) \div \left( \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A} \right)$   
 $= (\cos^2 A - \sin^2 A) \div (\cos^2 A + \sin^2 A)$   
 $= \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$   
 $= 1 - 2 \sin^2 A.$

$$4. \quad \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13};$$

$$\therefore \tan A + \sec A = \frac{5}{12} + \frac{13}{12} = \frac{3}{2}.$$

5. Let  $AB = 15$ ,  $AC = 30$ , then since  $\cos 60^\circ = \frac{1}{2}$ , it is easy to see that  $CB$  is at right angles to  $BA$ .

$$\therefore CB = CA \sin 60^\circ = 30 \times \frac{\sqrt{3}}{2} = 25.98.$$

6. Let  $AB$  be the tower,  $BD$  the cliff, and  $C$  the point of observation; then if  $BD = x$  ft.,  $CD = y$  ft., we have  $50 + x = y \tan \alpha$ ,  $x = y \tan \beta$ .

$$\text{By division} \quad 1 + \frac{50}{x} = \frac{\tan \alpha}{\tan \beta} = \frac{1260}{1185} = \frac{84}{79}.$$

$$\therefore \frac{50}{x} = \frac{5}{79}, \text{ and } x = 790.$$

7. If  $x$  ft. be the length of the arc,  $\frac{x}{30} = \text{radian measure of } 10^\circ$ ; whence

$$x = 30 \times \frac{\pi}{180} \times 10 = 5.236.$$

$$8. \quad \text{Here} \quad \tan \alpha = \frac{8}{15}; \quad \therefore \sin \alpha = \frac{8}{17}, \quad \cos \alpha = \frac{15}{17}.$$

$$9. \quad \text{Here} \quad 4 \sin^2 \theta - (2 + 2\sqrt{3}) \sin \theta + \sqrt{3} = 0;$$

$$\text{whence} \quad (2 \sin \theta - \sqrt{3})(2 \sin \theta - 1) = 0.$$

$$10. \quad \text{The expression} = \frac{5 \tan \alpha + 7}{6 - 3 \tan \alpha} = \frac{5 \times 4 + 7 \times 15}{6 \times 15 - 3 \times 4}$$

$$= \frac{5}{3} \times \frac{25}{26} = \frac{125}{78}.$$

$$11. \quad \text{First side} = 1 + 2(\sin A + \cos A) + (\sin A + \cos A)^2$$

$$= 1 + 2(\sin A + \cos A) + 1 + 2 \sin A \cos A$$

$$= 2 + 2(\sin A + \cos A) + 2 \sin A \cos A$$

$$= 2(1 + \sin A)(1 + \cos A).$$

$$12. \quad \text{The expression} = \sec^2 A (2 - \sec^2 A) - \operatorname{cosec}^2 A (2 - \operatorname{cosec}^2 A)$$

$$= (1 + \tan^2 A)(1 - \tan^2 A) - (1 + \cot^2 A)(1 - \cot^2 A)$$

$$= (1 - \tan^4 A) - (1 - \cot^4 A)$$

$$= (1 - \tan^4 A) - \left(1 - \frac{1}{\tan^4 A}\right)$$

$$= (1 - \tan^4 A) \left(1 + \frac{1}{\tan^4 A}\right) = \frac{1 - \tan^8 A}{\tan^4 A}.$$

$$13. \quad \sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\sin \alpha - \cos \alpha}{\sqrt{(\sin \alpha - \cos \alpha)^2 + (\sin \alpha + \cos \alpha)^2}}$$

$$= \frac{\sin \alpha - \cos \alpha}{\sqrt{2(\sin^2 \alpha + \cos^2 \alpha)}} = \frac{\sin \alpha - \cos \alpha}{\sqrt{2}}.$$

$$14. \quad \text{First side} = \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right) \div \left( \frac{1}{\sqrt{2}} + \frac{1}{2} \right) = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = \frac{\sec 45^\circ - \tan 45^\circ}{\operatorname{cosec} 45^\circ + \cot 45^\circ}.$$

$$15. \quad \text{Since } 9 \text{ degrees} = 10 \text{ grades}$$

$$1' = \frac{1}{60} \times \frac{10}{9} \times 100 \times 100 \times 1'' = \frac{10^4}{54} \times 1''.$$

$$16. \quad (2) \quad \text{Second side} = \left( 1 - \frac{\sin \theta}{\cos \theta} \right)^2 \div \left( 1 - \frac{\cos \theta}{\sin \theta} \right)^2$$

$$= \left( \frac{\cos \theta - \sin \theta}{\cos \theta} \right)^2 \div \left( \frac{\sin \theta - \cos \theta}{\sin \theta} \right)^2$$

$$= \frac{\sec^2 \theta}{\operatorname{cosec}^2 \theta} = \frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}.$$

$$17. \quad (1) \quad \sin \theta + \frac{1}{\sin \theta} = \frac{3}{\sqrt{2}};$$

$$\therefore \sin^2 \theta - \frac{3}{\sqrt{2}} \sin \theta + 1 = 0,$$

$$\sqrt{2} \sin^2 \theta - 3 \sin \theta + \sqrt{2} = 0,$$

$$(\sqrt{2} \sin \theta - 1)(\sin \theta - \sqrt{2}) = 0;$$

whence  $\sin \theta = \frac{1}{\sqrt{2}}$ , and  $\theta = 45^\circ$ , the other value being impossible.

$$(2) \quad \cos \theta + \frac{1}{\cos \theta} = \frac{5}{2};$$

$$\therefore 2 \cos^2 \theta - 5 \cos \theta + 2 = 0,$$

$$(2 \cos \theta - 1)(\cos \theta - 2) = 0;$$

whence  $\cos \theta = \frac{1}{2}$ , and  $\theta = 60^\circ$ , the other value being impossible.

$$18. \quad \text{Radian measure of } 56^\circ = \frac{\pi}{180} \times 56 = \frac{22}{7} \times \frac{56}{180} = \frac{44}{45}.$$

$$\text{The arc traversed in } 36'' = \frac{36}{60} \times \frac{10}{60} \times 1760 \text{ yards} = 176 \text{ yards.}$$

$\therefore$  if  $d$  be the number of yards in the diameter,

$$\frac{176}{d} = \frac{22}{45}; \quad \therefore d = 360.$$

19. See Art. 35.

$$20. (1) \text{ First side} = 1 \times \left( \frac{\sin^2 A}{\cos^2 A} - 1 \right) = \frac{\sin^2 A - \cos^2 A}{\cos^2 A} \\ = \sec^2 A (\sin^2 A - \cos^2 A) = \text{second side.}$$

$$(2) \text{ Second side} = \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \beta} - \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha} \\ = \sin^2 \alpha - \sin^2 \beta \\ = \text{First side.} \quad [\text{Examples III. b, 34.}]$$

$$21. \text{ We have } \cos B = \frac{a}{c} = .405; \text{ and } a + c = 281.$$

$$\therefore c(1 + .405) = 281; \text{ whence } c = 200, a = 81.$$

$$\text{Also } b = \sqrt{c^2 - a^2} = \sqrt{33439} = 183 \text{ nearly.}$$

$$22. \text{ The radian measure} = \frac{\text{arc}}{\text{radius}} = \frac{495}{3 \times 1760} = \frac{3}{32} = .09375.$$

$$\text{With this as unit a right angle would be } \frac{\pi}{2} \div \frac{3}{32} = 16.7552.$$

$$23. (1) \text{ First side} = \sin \theta \cos \theta \left\{ \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \right\} = \cos^2 \theta + \sin^2 \theta = 1.$$

$$(2) \text{ First side} = \frac{\cot \theta}{\sec \theta} \times \frac{\cot^2 \theta}{\operatorname{cosec} \theta} \times \frac{\cos \theta}{\sin^3 \theta} \\ = \cot^3 \theta \times \frac{\cos \theta}{\sin \theta} \times \frac{\cos \theta}{\sin \theta} = \cot^5 \theta.$$

24. If  $O$  be the point of observation,  $\angle BOA = 45^\circ$ , and the line drawn from  $B$  perpendicular to  $OA$  bisects it at a point  $A'$ . Then

$$OA : OB = 2OA' : OB = 2 \cos 45^\circ = \sqrt{2} : 1.$$

$$25. (1) \text{ First side} = (\sin^2 A + \operatorname{cosec}^2 A - 2) + (\cos^2 A + \sec^2 A - 2) \\ = (\sin^2 A + \cos^2 A) + (\operatorname{cosec}^2 A - 1) + (\sec^2 A - 1) - 2 \\ = \cot^2 A + \tan^2 A - 1.$$

$$(2) \text{ First side} = 3 \cot^2 \theta - 10 \cot \theta + 3 = 3 \operatorname{cosec}^2 \theta - 10 \cot \theta.$$

$$27. \text{ The expression} = \frac{2 - \cot A}{2 + 3 \cot A} = \frac{2 - \frac{9}{2}}{2 + \frac{27}{2}} = -\frac{5}{31}.$$

28.  $2 \cos \theta \cot \theta + 1 - \cot \theta - 2 \cos \theta = 0;$

$$\therefore 2 \cos \theta (\cot \theta - 1) - (\cot \theta - 1) = 0;$$

$$\therefore (2 \cos \theta - 1) (\cot \theta - 1) = 0;$$

$$\therefore \theta = 60^\circ, \text{ or } 45^\circ.$$

29. If  $x$  inches be the length of the arc,  $\frac{x}{6} = \frac{\pi}{180} \times \frac{1217}{60}.$

$$\text{In the second circle } \theta = \frac{\pi}{180} \times \frac{1217}{10} \times \frac{1}{8};$$

$$\therefore \text{sexagesimal measure of } \theta = \frac{1217}{10 \times 8} = 15^\circ 12' 45''.$$

30. Take the figure of the Example on p. 41, and let

$$PT = x \text{ yards, } RT = y \text{ yards.}$$

Then  $x = y + 110$ , since  $\angle PQT = 45^\circ.$

Also

$$y = x \cot 60^\circ = \frac{x}{\sqrt{3}}.$$

$$\therefore x \left( \frac{\sqrt{3} - 1}{\sqrt{3}} \right) = 110;$$

$$\therefore x = 55 (3 + \sqrt{3}) = 260.26.$$

$$\begin{aligned} 31. \text{ First side} &= \frac{1 + \cos A}{1 - \cos A} - \frac{1 - \cos A}{1 + \cos A} - 4 \cot^2 A \\ &= \frac{(1 + \cos A)^2 - (1 - \cos A)^2}{(1 - \cos A)(1 + \cos A)} - 4 \cot^2 A \\ &= \frac{2 \cdot 2 \cos A}{\sin^2 A} - \frac{4 \cos^2 A}{\sin^2 A} = \frac{4 \cos A (1 - \cos A)}{1 - \cos^2 A} \\ &= \frac{4 \cos A}{1 + \cos A} = \frac{4}{1 + \sec A}. \end{aligned}$$

32. (1)  $8(1 - \cos^2 \theta) - 2 \cos \theta = 5;$

$$8 \cos^2 \theta + 2 \cos \theta - 3 = 0;$$

$$(2 \cos \theta - 1)(4 \cos \theta + 3) = 0;$$

whence  $\theta = 60^\circ$ ; or  $\cos \theta = -\frac{3}{4}.$

(2)  $5 \tan^2 x - (1 + \tan^2 x) = 11.$

$$4 \tan^2 x = 12;$$

$$\tan x = \pm \sqrt{3}.$$

From the first of these values  $x = 60^\circ.$

33. Here  $\theta = \frac{\text{arc}}{\text{radius}} = \frac{11}{5 \times 12};$

$$\therefore \text{the angle in degrees} = \frac{180}{\pi} \times \frac{11}{60} = \frac{180}{22} \times \frac{11 \times 7}{60} = 10\frac{1}{2}^\circ.$$

35. See Art. 16.

(2) Here  $\cos \theta = \frac{2ab}{a^2 + b^2}.$

Now since  $(a - b)^2$  is a positive quantity,  $a^2 + b^2 > 2ab$ ;  $\therefore \cos \theta < 1$ , which is possible.

36. Let  $ACB$  be the hill,  $A$  being the summit and  $C$  a point halfway down. Draw  $AD$ ,  $CE$  perpendicular to the horizontal line through the object  $O$ . Then  $AD = 2CE$ , and  $BD = 2BE$ .

Now  $OB + BE = CE \cot \beta$ , and  $OB + 2BE = 2CE \cot \alpha$ ;

therefore, by subtraction,  $\frac{OB + 2BE}{CE} - \frac{OB + BE}{CE} = 2 \cot \alpha - \cot \beta$ ;

that is,  $\frac{BE}{CE} = 2 \cot \alpha - \cot \beta$ ; also  $\frac{BE}{CE} = \cot \theta$ .

37. From the figure in Ex. 2, Art. 46, we have

$$\cos B = \frac{a}{c} = \frac{25\sqrt{2}}{50} = \frac{1}{\sqrt{2}}; \therefore B = 45^\circ;$$

$$\therefore b = c \sin 45^\circ = 50 \times \frac{1}{\sqrt{2}} = 25\sqrt{2}.$$

If  $p$  be the perpendicular from  $C$ ,  $p = a \sin 45^\circ = 25$ .

40. Since  $\tan \theta = \pm 1$ , the angles will be those coterminal with  $45^\circ, 135^\circ, 225^\circ, 315^\circ$ .

41. Take the figure of Example II. on p. 43.

Let  $AE = x$ ,  $CE = y$ ; then

$$\frac{x}{y} = .965; \quad \frac{x + 42}{y} = 1.6.$$

$$\therefore \frac{x + 42}{x} = \frac{1600}{965} = 1 + \frac{635}{965};$$

whence

$$x = 42 \times \frac{965}{635} = 63, \text{ approximately};$$

$$\therefore AB = 63 + 42 = 105.$$

$$42. \text{ We have } \tan 15^\circ = \frac{\sin 30^\circ}{1 + \cos 30^\circ} = \frac{1}{2} / \left(1 + \frac{\sqrt{3}}{2}\right) = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}.$$

Similarly

$$\tan 75^\circ = 2 + \sqrt{3}.$$

Again,

$$1 + \tan^2 \theta = 4 \tan \theta,$$

$$\tan^2 \theta - 4 \tan \theta + 1 = 0;$$

$$\therefore \tan \theta = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3};$$

$$\therefore \theta = 75^\circ, \text{ or } 15^\circ.$$

43. The first side

$$= 1 + \sec \theta + \tan \theta + \operatorname{cosec} \theta + \sec \theta \operatorname{cosec} \theta$$

$$+ \operatorname{cosec} \theta \tan \theta + \cot \theta + \sec \theta \cot \theta + 1$$

$$= 2 + \sec \theta + \tan \theta + \operatorname{cosec} \theta + \sec \theta \operatorname{cosec} \theta + \sec \theta + \cot \theta + \operatorname{cosec} \theta$$

$$= 2(1 + \sec \theta + \operatorname{cosec} \theta) + \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta \cos \theta}$$

$$= 2(1 + \sec \theta + \operatorname{cosec} \theta) + \frac{2(\sin^2 \theta + \cos^2 \theta)}{\sin \theta \cos \theta}$$

$$= 2(1 + \sec \theta + \operatorname{cosec} \theta + \tan \theta + \cot \theta).$$

44. See figure and notation of Art. 25.

$$45. \text{ We have } 2(1 - \sin^2 \theta) = 1 + \sin \theta;$$

$$\therefore (1 + \sin \theta)(1 - 2 \sin \theta) = 0;$$

$$\therefore \sin \theta = -1, \text{ or } \sin \theta = \frac{1}{2};$$

whence

$$\theta = 30^\circ, 150^\circ, 270^\circ.$$

$$46 \quad \sin(270^\circ + A) = -\sin(90^\circ + A) = -\cos A.$$

But

$$\cos A = \pm \sqrt{1 - \sin^2 A} = \pm .8;$$

$$\therefore \sin(270^\circ + A) = \pm .8.$$

47. See Art. 113.

$$48. \text{ The expression } = \frac{2 \sin 2A \sin A}{2 \sin A \cos 2A} = \tan 2A. \text{ See Art. 89.}$$

$$49. \tan A = \sqrt{\sec^2 A - 1} = \pm \sqrt{\frac{4}{3} - 1} = \pm \frac{1}{\sqrt{3}}.$$

The angle is coterminal with  $150^\circ$  or  $210^\circ$ , and the tangents of these angles are equal but opposite in sign.

50. Take the figure of Example on p. 41, and let  
 $PT = x$  yards,  $\angle PQT = 45^\circ$ ,  $\angle PRT = 60^\circ$ .

Also  $QR = 1760$ , and  $QT = PT$ ;

$$\therefore \frac{x}{x - 1760} = \tan 60^\circ = \sqrt{3};$$

$$\therefore x(\sqrt{3} - 1) = 1760\sqrt{3};$$

$$\therefore x = 880(3 + \sqrt{3}) = 4164.16.$$

52. (1) First side  $= \sin^2 a (\sin^2 a + 2 \cos^2 a)$   
 $= (1 - \cos^2 a)(1 + \cos^2 a) = 1 - \cos^4 a.$

$$(2) \text{ First side} = \sec 2 \left( \frac{\pi}{4} - \theta \right) \quad [\text{Art. 124}]$$

$$= \operatorname{cosec} 2\theta.$$

$$(3) \cos 10^\circ + \sin 40^\circ = \cos 10^\circ + \cos 50^\circ = 2 \cos 30^\circ \cos 20^\circ$$

$$= \sqrt{3} \sin 70^\circ.$$

$$53. \text{ The expression} = \frac{a \tan \theta - b}{a \tan \theta + b} = \frac{a^2 - b^2}{a^2 + b^2}.$$

54. Multiply all through by  $\cos 18^\circ$ , then we have to prove that  
 $4 \cos^2 18^\circ - 3 = 2 \sin 18^\circ.$

$$\text{First side} = 4 \left\{ 1 - \left( \frac{\sqrt{5} - 1}{4} \right)^2 \right\} - 3 = \frac{10 + 2\sqrt{5}}{4} - 3$$

$$= \frac{\sqrt{5} - 1}{2} = 2 \sin 18^\circ.$$

$$56. \text{ Here } \theta = \frac{\text{arc}}{\text{radius}} = \frac{20 \times 10}{60 \times 60} \div \frac{1}{2} = \frac{1}{9};$$

$$\therefore D = \frac{1}{9} \times 180 \times \frac{7}{22} = \frac{70}{11} = 6 \frac{4}{11}^\circ.$$

57. The expression  $= \frac{2 - 3 \cot a}{4 - 9 \tan a}$ , where  $\tan a = \frac{12}{5}$ ,

$$= \frac{2 - 3 \times \frac{5}{12}}{4 - 9 \times \frac{12}{5}} = \frac{3}{4} \div \left( -\frac{88}{5} \right)$$

$$= -\frac{15}{352}.$$

$$\begin{aligned}
 58. \quad (2) \quad \text{First side} &= \frac{2 \sin \frac{\pi}{4} \sin \theta}{2 \cos \frac{2\pi}{3} \sin \theta} + \sqrt{2} \\
 &= \frac{1}{\sqrt{2}} \left( -\frac{1}{2} \right) + \sqrt{2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 59. \quad (1) \quad \text{The expression} &= \frac{\cos A \cos C + (-\cos C)(-\cos A)}{\cos A \sin C - \sin C(-\cos A)} \\
 &= \frac{2 \cos A \cos C}{2 \cos A \sin C} = \cot C.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{The expression} &= \frac{\sin A \cos A + \text{two similar terms}}{\sin A \sin B \sin C} \\
 &= \frac{1}{2} \left( \frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C} \right) \\
 &= 2. \quad [\text{Art. 135, Ex. 1.}]
 \end{aligned}$$

60. Let  $ABC$  be the horizontal equilateral triangle, and let  $PQ$  be the flagstaff. Then since each side subtends an angle of  $60^\circ$  at  $P$ , the top of the flagstaff, the triangles  $PCB$ ,  $PBA$ ,  $PAC$  are equilateral.

Let  $x$  be a side of  $\triangle ABC$ ;

then  $AQ = \frac{x}{2 \sec 30^\circ} = \frac{x}{\sqrt{3}}.$

Then from  $\triangle PAQ$ ,

$$PA^2 = AQ^2 + QP^2,$$

$$x^2 = \frac{x^2}{3} + 10000;$$

whence

$$x^2 = 15000, \text{ or } x = 50\sqrt{6}.$$

$$\begin{aligned}
 61. \quad \text{The first expression} &= \sin \theta + \cos \theta + \sin^2 \theta + \cos^2 \theta \\
 &= \sin \theta + \cos \theta + 1.
 \end{aligned}$$

Similarly, the second expression  $= \sin \theta + \cos \theta - 1$ ;

$$\therefore \text{the product} = (\sin \theta + \cos \theta)^2 - 1 = 2 \sin \theta \cos \theta = \sin 2\theta.$$

$$\begin{aligned}
 62. \quad \text{Second side} &= \sin^2 \frac{\theta + \phi}{2} + \cos^2 \frac{\theta - \phi}{2} - 2 \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\
 &= \frac{1}{2} \{ 1 - \cos (\theta + \phi) + 1 + \cos (\theta - \phi) \} - \sin \theta - \sin \phi. \\
 &= 1 - \sin \theta - \sin \phi + \frac{1}{2} \{ \cos (\theta - \phi) - \cos (\theta + \phi) \} \\
 &= 1 - \sin \theta - \sin \phi + \sin \theta \sin \phi \\
 &= (1 - \sin \theta)(1 - \sin \phi).
 \end{aligned}$$

63. The expression  $= \frac{2 \cos \alpha \sin \theta}{2 \sin \beta \sin \theta} = \frac{\cos \alpha}{\sin \beta}$ , which is independent of  $\theta$ .

64. First side  $= \sin \frac{B+C}{2} \cos \frac{B-C}{2} + \dots + \dots$

$$= \frac{1}{2} \{ (\sin B + \sin C) + \dots + \dots \}$$

$$= \sin A + \sin B + \sin C.$$

65. We have  $\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}};$

$$2 \sin^2 \frac{\theta}{2} = \left( 2 \cos^2 \frac{\theta}{2} - 1 \right)^2;$$

$$\therefore 4 \cos^4 \frac{\theta}{2} - 2 \cos \frac{\theta}{2} - 1 = 0;$$

$$\therefore \cos^2 \frac{\theta}{2} = \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4};$$

$$\therefore \cos^2 \frac{\theta}{2} = \cos 36^\circ, \text{ the other value being impossible.}$$

66. Draw  $ZW$  perpendicular to  $XY$  and let  $ZW = x$ . Then  $\triangle XZY$  right-angled at  $Z$ .

And  $XZ = XY \cos 30^\circ = 100 \sqrt{3}$  yds.

Again, from  $\triangle WXZ$ ,  $WZ = XZ \sin 30^\circ = 50 \sqrt{3} = 86.6$  yds.

67. We have  $32\pi \times 1000 = \frac{5585}{2} \times 3 \times 12$ ; whence  $\pi = 3.141$ , approximately.

$$68. \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{1 - \cos 2\alpha}{2} + \dots + \dots$$

$$= \frac{3}{2} - \frac{1}{2} (\cos 2\alpha + \cos 2\beta + \cos 2\gamma)$$

$$= \frac{3}{2} - \frac{1}{2} \{ 2 \cos (\alpha + \beta) \cos (\alpha - \beta) + 1 - 2 \sin^2 \gamma \}$$

$$= 1 - \{ \sin \gamma \cos (\alpha - \beta) - \sin^2 \gamma \}$$

$$= 1 - \sin \gamma \{ \cos (\alpha - \beta) - \cos (\alpha + \beta) \}$$

$$= 1 - 2 \sin \alpha \sin \beta \sin \gamma.$$

$$69. \quad (1) \quad \text{First side} = \left( \frac{\sin A}{\cos A} + \frac{\sin 2A}{\cos 2A} \right) (2 \cos 2A \cos A) \\ = 2 (\sin 2A \cos A + \cos 2A \sin A) = 2 \sin 3A.$$

(2) Multiply all through by 32; then

$$\begin{aligned} \text{Second side} &= 2 + \cos 2A - 2 \cos 4A - \cos 6A \\ &= 2 (1 - \cos 4A) + 2 \sin 4A \sin 2A \\ &= 4 \sin^2 2A + 4 \sin^2 2A \cos 2A \\ &= 4 \sin^2 2A (1 + \cos 2A) \\ &= 16 \sin^2 A \cos^2 A \cdot 2 \cos^2 A \\ &= 32 \sin^2 A \cos^4 A. \end{aligned}$$

$$70. \quad \text{The expression} = \frac{2 \cos 13a \sin 10a}{2 \sin 10a \cos 6a} = \frac{\cos 13a}{\cos 6a} \\ = \frac{\cos 13a}{\cos (\pi - 13a)} = -1.$$

71. We have  $\cot (A+B) = 1$ . Therefore

$$\begin{aligned} \cot A \cot B - 1 &= \cot A + \cot B; \\ \therefore 2 \cot A \cot B &= 1 + \cot A + \cot B + \cot A \cot B \\ &= (1 + \cot A) (1 + \cot B); \\ \therefore \frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} &= \frac{1}{2}. \end{aligned}$$

72. For the first part see XI. d. Ex. 15.

Then

$$\begin{aligned} \tan \theta &= \cot \theta - 2 \cot 2\theta, \\ 2 \tan 2\theta &= 2 \cot 2\theta - 4 \cot 4\theta, \\ 4 \tan 4\theta &= 4 \cot 4\theta - 8 \cot 8\theta; \end{aligned}$$

$\therefore$  by addition

$$\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta = \cot \theta - 8 \cot 8\theta.$$

$$\begin{aligned} 73. \quad \text{The expression} &= 1 - \frac{\sin^3 \theta}{\sin \theta + \cos \theta} - \frac{\cos^3 \theta}{\sin \theta + \cos \theta} \\ &= 1 - \frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} \\ &= 1 - (\sin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta) \\ &= \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta. \end{aligned}$$

74. We have  $x = 3 \sin A - (3 \sin A - 4 \sin^3 A) = 4 \sin^3 A$ ,  
 $y = 4 \cos^3 A - 3 \cos A + 3 \cos A = 4 \cos^3 A$ ;

$$\therefore \left(\frac{x}{4}\right)^{\frac{2}{3}} + \left(\frac{y}{4}\right)^{\frac{2}{3}} = \sin^2 A + \cos^2 A = 1;$$

$$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}.$$

75. Let  $PQ, RS$  be the flagstaffs of lengths  $x, y$  feet respectively. Then  $QABS$  is a straight line, such that  $AB = 30$  ft.,  $\angle PAQ = 60^\circ$ ,  $\angle RAS = 30^\circ$ ,  $\angle PBQ = 45^\circ$ ,  $\angle RBS = 60^\circ$ . Let  $AQ = a$ ,  $BS = b$ . Then since

$$\angle PBQ = 45^\circ, BQ = QP = x;$$

$$\therefore a = x - 30.$$

From  $\triangle ARS$ , we have  $AS = RS \cot 30^\circ = y \sqrt{3}$ ;

$$\therefore b = y \sqrt{3} - 30.$$

From  $\triangle APQ$ ,  $PQ = AQ \tan 60^\circ$ ;

$$\therefore x = (x - 30) \sqrt{3};$$

$$x = 15(3 + \sqrt{3}).$$

whence

From  $\triangle BRS$ ,  $RS = BS \tan 60^\circ$ ;

$$\therefore y = \sqrt{3}(y \sqrt{3} - 30);$$

$$y = 15 \sqrt{3}.$$

whence

Again,

$$QS = a + b + 30$$

$$= x + y \sqrt{3} - 30$$

$$= 45 + 15 \sqrt{3} + 15 - 30$$

$$= 60 + 15 \sqrt{3}.$$

76. (1) First side

$$= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) - 2 \cos A \cos B \cos (A + B)$$

$$= 1 + \cos (A + B) \cos (A - B) - 2 \cos A \cos B \cos (A + B)$$

$$= 1 + \cos (A + B) \{ \sin A \sin B - \cos A \cos B \}$$

$$= 1 - \cos^2 (A + B) = \sin^2 (A + B).$$

(2) First side

$$= 2(\sin 5A - \sin A) - (\sin 3A + \sin A)$$

$$= 4 \cos 3A \sin 2A - 2 \sin 2A \cos A$$

$$= 2 \sin 2A (2 \cos 3A - \cos A)$$

$$= 4 \sin A \cos A (8 \cos^3 A - 7 \cos A)$$

$$= 4 \sin A \cos^2 A \{ 8(1 - \sin^2 A) - 7 \}$$

$$= 4 \sin A \cos^2 A (1 - 8 \sin^2 A).$$

77. If  $r$  is the radius of the circle, and  $AB$  one side of the square, we have  $2\pi r = 3$ , and

$$\begin{aligned} AB &= 2r \sin 45^\circ = \frac{3\sqrt{2}}{2\pi} \\ &= \frac{3}{2} \times 1.4142 \times .3183 \\ &= .6752 \text{ feet} \\ &= 8.10 \text{ inches.} \end{aligned}$$

78. Here  $AB = 2r \sin 54^\circ$ ,  $BC = 2r \sin 30^\circ$ ,  $CD = 2r \sin 18^\circ$ . And it remains to prove that

$$\sin 54^\circ = \sin 30^\circ + \sin 18^\circ.$$

[See Examples XI. c. 9.]

79. First side  $= (2 + \sqrt{3}) + (2 - \sqrt{3}) - 1 - 2 = 1$ .

80. We have, by addition,  $\cot \theta = 2(m + n)$ .

Also, by subtraction,  $\cos \theta = 2(m - n)$ .

$$\therefore 4(m^2 - n^2) = \frac{\cos^2 \theta}{\sin \theta};$$

$$\begin{aligned} \therefore 16(m^2 - n^2)^2 &= \frac{\cos^4 \theta}{\sin^2 \theta} = \cot^2 \theta \times \cos^2 \theta \\ &= \cot^2 \theta (1 - \sin^2 \theta) \\ &= 16mn. \end{aligned}$$

81. (1) First side  $= \frac{1}{2} [\sin (2\beta + \alpha) + \sin \alpha - \sin (2\gamma + \alpha) - \sin \alpha]$   
 $= \frac{1}{2} [2 \cos (\alpha + \beta + \gamma) \sin (\beta - \gamma)].$

(2) First side  $= \left( \frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A} \right) \left( \frac{1}{\cos A} + \frac{1}{\cos 3A} \right)$   
 $= \frac{\sin A}{\cos 2A \cos A} \cdot \frac{\cos 3A + \cos A}{\cos A \cos 3A}$   
 $= \frac{2 \sin A \cos 2A \cos A}{\cos 2A \cos^2 A \cos 3A} = 2 \sin A \sec A \sec 3A.$

82.  $2 \cos 6^\circ \cos 66^\circ = \cos 72^\circ + \cos 60^\circ = \sin 18^\circ + \frac{1}{2}$   
 $= \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4}.$

$$2 \cos 42^\circ \cos 78^\circ = \cos 120^\circ + \cos 36^\circ$$

$$= -\frac{1}{2} + \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}-1}{4}.$$

$$\therefore 4 \cos 6^\circ \cos 66^\circ \cos 42^\circ \cos 78^\circ = \frac{1}{4}.$$

83. Put  $2A = 45^\circ$ .

84. The distance required is evidently equal to  $10 \tan 22\frac{1}{2}^\circ$

$$= \frac{10}{\sqrt{2}+1} = 10(\sqrt{2}-1) = 4.14 \text{ miles.}$$

85. (1) Separate each term into the difference of two cosines

$$\begin{aligned} (2) \text{ Second side} &= \frac{\sin \theta + 2 \sin \theta \cos \theta}{\cos \theta + 2 \cos^2 \theta} \\ &= \frac{\sin \theta (1 + 2 \cos \theta)}{\cos \theta (1 + 2 \cos \theta)} = \tan \theta. \end{aligned}$$

86. First side  $= 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2} - 1$

$$= 2 \cos \frac{\gamma}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2} - 1$$

$$= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha-\beta}{2} + \cos \frac{\gamma}{2} \right\} - 1$$

$$= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2} \right\} - 1$$

$$= 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} - 1.$$

87. Put  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ , then

$$\text{first side} = \frac{k^2 (\sin^2 B - \sin^2 C)}{k \sin A} \cos A + \dots + \dots$$

$$= k \left\{ \frac{\sin (B+C) \sin (B-C)}{\sin A} \cos A + \dots + \dots \right\}$$

$$= k \{ \sin (B-C) \cos A + \dots + \dots \}$$

$$= -k \{ \sin (B-C) \cos (B+C) + \dots + \dots \}$$

$$= -\frac{k}{2} \{ (\sin 2B - \sin 2C) + \dots + \dots \}$$

$$= 0.$$

$$\begin{aligned}
 88. \quad \text{First side} &= a \cos 2\theta + b \sin 2\theta \\
 &= a(1 - 2 \sin^2 \theta) + 2b \sin \theta \cos \theta \\
 &= a + 2 \sin \theta (b \cos \theta - a \sin \theta) \\
 &= a, \text{ since } b \cos \theta = a \sin \theta.
 \end{aligned}$$

$$\begin{aligned}
 89. \quad \text{Let } \log_a b &= x, \text{ so that } a^x = b, \\
 \log_b c &= y, \dots\dots\dots b^y = c, \\
 \log_c a &= z, \dots\dots\dots c^z = a.
 \end{aligned}$$

$$\begin{aligned}
 \text{Then we have} \quad a &= c^z = b^{yz} = a^{xyz}; \\
 \therefore xyz &= 1, \text{ or } \log_a b \log_b c \log_c a = 1.
 \end{aligned}$$

We have  $\log 8 = \log 2^3 = 3 \log 2$ ; whence  $\log 2 = .30103$ ;

$$\begin{aligned}
 \log 2.4 &= \log \left( \frac{3 \times 8}{10} \right) = \log 3 + \log 8 - 1 \\
 &= 1.47712 + .90309 = .38021.
 \end{aligned}$$

$$\begin{aligned}
 \log 5400 &= 2 + \log 2 + 3 \log 3 \\
 &= 2.30103 + 1.43136 = 3.73239.
 \end{aligned}$$

$$\begin{aligned}
 L \tan 30^\circ &= 10 + \log \frac{1}{\sqrt{3}} = 10 - \frac{1}{2} \log 3 \\
 &= 9.76144.
 \end{aligned}$$

$$\begin{aligned}
 90. \quad \cot (A + B) &= \cot (90^\circ - C) = \tan C = \frac{1}{\cot C}; \\
 \therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} &= \frac{1}{\cot C};
 \end{aligned}$$

whence by multiplying up and rearranging we obtain the required result.

For the second part, put  $A = 15^\circ$ ,  $B = 30^\circ$ ,  $C = 45^\circ$ .

$$\begin{aligned}
 91. \quad \text{First side} &= (1 + \sin 2A)^2 + \cos^2 2A + 2 \cos 2A (1 + \sin 2A) \\
 &= (1 + \sin 2A)^2 + (1 - \sin^2 2A) + 2 \cos 2A (1 + \sin 2A) \\
 &= (1 + \sin 2A) \{1 + \sin 2A + 1 - \sin 2A + 2 \cos 2A\} \\
 &= 2(1 + \cos 2A)(1 + \sin 2A) = 4 \cos^2 A (1 + \sin 2A).
 \end{aligned}$$

92. See Art. 150.

93. We have to prove that  $\sin 9^\circ \sin 81^\circ = \sin 12^\circ \sin 48^\circ$ .

$$\begin{aligned}
 \text{First side} &= \frac{1}{2} (\cos 72^\circ - \cos 90^\circ) \\
 &= \frac{1}{2} \sin 18^\circ = \frac{\sqrt{5} - 1}{8}.
 \end{aligned}$$

$$\begin{aligned}\text{Second side} &= \frac{1}{2} (\cos 36^\circ - \cos 60^\circ) \\ &= \frac{1}{2} \left( \frac{\sqrt{5}+1}{4} - \frac{1}{2} \right) = \frac{\sqrt{5}-1}{8}.\end{aligned}$$

94. See Art. 136, Ex. 2.

95.  $L \sin \theta > L \sin 27^\circ 45'$  by  $\frac{1742}{2400} \times 60''$ ;

whence  $\theta = 27^\circ 45' 44''$ .

96. By a well-known algebraical formula,

$$x^3 + y^3 + z^3 = 3xyz,$$

when

$$x + y + z = 0;$$

therefore we have

$$\cos^3 A + \cos^3 B + \cos^3 C = 3 \cos A \cos B \cos C.$$

Substituting  $\frac{1}{4}(\cos 3A + 3 \cos A)$  for  $\cos^3 A$ , and similar results for  $\cos^3 B$ ,  $\cos^3 C$

we have

$$\frac{1}{4}(\cos 3A + \cos 3B + \cos 3C) + \frac{3}{4}(\cos A + \cos B + \cos C) = 3 \cos A \cos B \cos C$$

whence the required result follows at once, since

$$\cos A + \cos B + \cos C = 0.$$

97.  $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{49}{625}} = \pm \frac{24}{25}$ ; but since  $A$  lies between  $270^\circ$  and  $360^\circ$ , we must reject the negative value; thus  $\cos A = \frac{24}{25}$ .

$$\begin{aligned}\text{Hence } \sin 2A &= 2 \sin A \cos A = 2 \left( -\frac{7}{25} \right) \left( \frac{24}{25} \right) \\ &= -\frac{336}{625}.\end{aligned}$$

$$\begin{aligned}\text{Also } \tan \frac{A}{2} &= \frac{1 - \cos A}{\sin A} = \operatorname{cosec} A - \cot A \\ &= -\frac{25}{7} + \frac{24}{7} = -\frac{1}{7}.\end{aligned}$$

98. We have 
$$\frac{2 \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \left( \frac{\sin \theta + \cos \theta}{\sin \theta} \right)^2;$$

$$8 \cos^3 \frac{\theta}{2} \sin \frac{\theta}{2} = 1 + \sin 2\theta,$$

$$4 \cos^2 \frac{\theta}{2} \sin \theta = 1 + \sin 2\theta,$$

$$2 \sin \theta (1 + \cos \theta) = 1 + \sin 2\theta;$$

$$\therefore 2 \sin \theta = 1,$$

or

$$\theta = 30^\circ.$$

Again, by putting  $\theta = 30^\circ$ , we have

$$2 \cot 15^\circ = (1 + \sqrt{3})^2 = 4 + 2\sqrt{3}.$$

$$\therefore \cot 15^\circ = 2 + \sqrt{3}, \text{ and } \tan 15^\circ = 2 - \sqrt{3}.$$

99. We have  $\log 360 = 2 \log 2 + 2 \log 3 + 1.$

$$\therefore 2 \log 3 = \log 360 - 2 \log 2 - 1$$

$$= 1.5563025 - .6020600$$

$$= .9542425;$$

$$\therefore \log 3 = .4771213.$$

Now

$$\log .04 = \log 4 - 2 = 2 \log 2 - 2 = \bar{2}.60206.$$

$$\log 24 = 3 \log 2 + \log 3 = .90309 + .4771213$$

$$= 1.3802113.$$

$$\log 6 = \log \frac{2}{3} = \log 2 - \log 3$$

$$= .30103 - .4771213 = \bar{1}.8239087.$$

Again let  $\log_2 30 = x$ , so that  $2^x = 30$ .

$$\therefore x \log 2 = \log 30 = 1 + \log 3;$$

$$\therefore x = \frac{1 + \log 3}{\log 2} = \frac{1.4771213}{.30103} = 4.90689.$$

100. This follows at once from Art. 134, Ex. 5.

$$\begin{aligned}
 101. \cot(\alpha + \beta) &= \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta} = \frac{x(x + x^{-1} + 1) - 1}{(x + x^{-1} + 1)^{\frac{1}{2}}(1 + x)} = \frac{x^2 + x}{(1 + x)(x + x^{-1} + 1)^{\frac{1}{2}}} \\
 &= \frac{x}{(x + x^{-1} + 1)^{\frac{1}{2}}} = \frac{1}{x^{-1}(x + x^{-1} + 1)^{\frac{1}{2}}} = \frac{1}{(x^{-1} + x^{-3} + x^{-2})^{\frac{1}{2}}} \\
 &= \cot \gamma.
 \end{aligned}$$

Therefore  $\alpha + \beta = \gamma$ .

102. Let  $A$  and  $a$  be given, and let  $B$  be the right angle; then  $c = a \cot A$ ,  $b = a \operatorname{cosec} A$ , or  $b = \sqrt{a^2 + c^2}$ . Also  $C = 90^\circ - A$ .

If  $A = 31^\circ 53' 26.8''$ ,  $a = 28$ , we have

$$\begin{aligned}
 c &= 28 \cot A. \\
 \log c &= \log 28 + \log \cot 31^\circ 53' 26.8'', \\
 \log 28 &= 1.4471580 \\
 \log \cot 31^\circ 53' &= .2061805 \\
 \text{decrease for } 26.8'' & \quad \quad \quad 1258 \\
 \therefore \log c &= 1.6532127; \therefore c = 45.
 \end{aligned}$$

Again

$$\begin{aligned}
 b^2 &= a^2 + c^2 = 2025 + 784 = 2809; \\
 \therefore b &= 53; \text{ also } C = 90^\circ - A = 58^\circ 6' 33.2''.
 \end{aligned}$$

104. The greatest angle,  $C$ , is opposite to  $\sqrt{x^2 + xy + y^2}$ .

$$\begin{aligned}
 \therefore \cos C &= \frac{x^2 + y^2 - (x^2 + xy + y^2)}{2xy} = -\frac{1}{2}; \\
 \therefore C &= 120^\circ.
 \end{aligned}$$

$$\begin{aligned}
 105. \cos 3A + \sin 3A &= 4 \cos^3 A - 3 \cos A + 3 \sin A - 4 \sin^3 A \\
 &= 4(\cos^3 A - \sin^3 A) - 3(\cos A - \sin A),
 \end{aligned}$$

which is evidently divisible by  $\cos A - \sin A$ .

See solution to Examples XII. c, 27.

$$106. \text{ We have } \cot \frac{C}{2} = \tan \frac{A+B}{2} = \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5}{6} \cdot \frac{20}{37}} = \frac{5}{2}.$$

$$\therefore \tan C = \frac{2 \tan \frac{C}{2}}{1 - \tan^2 \frac{C}{2}} = \frac{2 \times \frac{5}{2}}{1 - \frac{25}{4}} = \frac{20}{21}.$$

For the second part, it will be sufficient to prove that

$$\sin A + \sin C = 2 \sin B.$$

$$\text{Now } \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{60}{61}, \text{ on reduction.}$$

$$\text{Similarly } \sin C = \frac{20}{29}, \text{ and } \sin B = \frac{1480}{1769}, \text{ whence the result follows.}$$

107. The first factor easily reduces to  $2 \cot 2\theta$ , and the second to  $\frac{2}{\cos 2\theta}$ ; whence the product becomes  $4 \operatorname{cosec} 2\theta$ .

$$108. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{128}{603}},$$

$$\log \tan \frac{A}{2} = \frac{1}{2} \{\log 128 - \log 603\} = \bar{1}.6634465;$$

$$\text{whence } \frac{A}{2} = 24^\circ 44' 16'', \text{ and } A = 49^\circ 28' 32''.$$

$$\begin{aligned} 109. \quad \tan(A-B) &= \frac{\tan A - \frac{n \sin A \cos A}{1 - n \sin^2 A}}{1 + \tan A \left( \frac{n \sin A \cos A}{1 - n \sin^2 A} \right)} = \frac{\tan A \left( 1 - \frac{n \cos^2 A}{1 - n \sin^2 A} \right)}{1 + \frac{n \sin^2 A}{1 - n \sin^2 A}} \\ &= (1-n) \tan A. \end{aligned}$$

$$110. \quad \log 200 = 2 + \log 2 = 3 - \log 5 = 2.30103.$$

$$\log .025 = 2 \log 5 - 3 = 2.39794,$$

$$\begin{aligned} \log \sqrt[3]{625} &= \frac{1}{3} (\log 625 - 1) = \frac{1}{3} (4 \log 5 - 1) \\ &= .598626. \end{aligned}$$

$$\begin{aligned} L \sin 30^\circ &= 10 + \log \left( \frac{1}{2} \right) = 10 + \log 5 - 1 \\ &= 9.69897. \end{aligned}$$

$$\begin{aligned} L \cos 45^\circ &= 10 + \log \left( \frac{1}{\sqrt{2}} \right) = 10 + \frac{1}{2} (\log 5 - 1) \\ &= 9.849485. \end{aligned}$$

$$\begin{aligned} 111. \quad (1) \quad \frac{\cot(A-30^\circ)}{\tan(A+30^\circ)} &= \frac{\cos(A-30^\circ) \cos(A+30^\circ)}{\sin(A-30^\circ) \sin(A+30^\circ)} \\ &= \frac{\cos 2A + \cos 60^\circ}{\cos 60^\circ - \cos 2A} = \frac{2 \cos 2A + 1}{1 - 2 \cos 2A} \\ &= \frac{2 + \sec 2A}{\sec 2A - 2}. \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{Second side} &= 2 \left\{ \frac{1 + \cos 2\alpha}{2} \cdot \frac{1 + \cos 2\beta}{2} + \frac{1 - \cos 2\alpha}{2} \cdot \frac{1 - \cos 2\beta}{2} \right\} \\
 &= \frac{1}{2} \{2 + 2 \cos 2\alpha \cos 2\beta\} = 1 + \cos 2\alpha \cos 2\beta.
 \end{aligned}$$

112. From a diagram it is easily seen that  $CD$  is equal to  $AC$ , and that from the right-angled triangle  $ABC$ ,

$$AC = BC \cos 30^\circ = 132 \frac{\sqrt{3}}{2}$$

$$= 66 \times \frac{19}{11} = 114 \text{ yards.}$$

Also the perp. from  $A$  on  $BC = AC \sin 30^\circ = 57$  yards.

$$113. \quad \text{Since } \frac{a}{a+b+c} = \frac{\sin A}{\sin A + \sin B + \sin C}$$

$$= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}, \quad [\text{XII. d. Ex. 3.}]$$

$$\begin{aligned}
 \therefore \frac{a \cot \frac{A}{2} + b \cot \frac{B}{2} - c \cot \frac{C}{2}}{a+b+c} &= \frac{1}{2} \left\{ \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \dots - \dots \right\} \\
 &= \frac{1}{2} \left\{ \frac{\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \right\}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} &= \frac{1}{2} \left\{ 2 \cos^2 \frac{A}{2} + 1 + \cos B - 1 - \cos C \right\} \\
 &= \frac{1}{2} \left\{ 2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} \sin \frac{C-B}{2} \right\} \\
 &= \cos \frac{A}{2} \left\{ \cos \frac{A}{2} + \sin \frac{C-B}{2} \right\} \\
 &= \cos \frac{A}{2} \left\{ \sin \frac{B+C}{2} + \sin \frac{C-B}{2} \right\} \\
 &= 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.
 \end{aligned}$$

Whence the required result easily follows.

H. E. T. K.

$$114. \text{ Here } \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{40 \times 42}{75 \times 77}}$$

$$= \frac{2}{\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{10}},$$

$$\log \cos \frac{A}{2} = \frac{3}{2} \log 2 - \frac{1}{2} \log 10$$

$$= .4515450 - .5$$

$$= \bar{1}.9515450$$

$$\log \cos 26^\circ 34' = \bar{1}.9515389$$

61

$$\therefore \frac{A}{2} \text{ is less than } 26^\circ 34' \text{ by } \frac{61}{632} \times 60''.$$

that is,

$$\frac{A}{2} = 26^\circ 33' 54.2'',$$

or

$$A = 53^\circ 7' 48''.$$

$$115. \sin (A - 90^\circ) = -\sin (90^\circ - A) = -\cos A$$

$$= -\sqrt{1 - \sin^2 A} = -\sqrt{6.4}$$

$$= -(\pm .8);$$

but  $A$  is between  $90^\circ$  and  $180^\circ$ , therefore  $\sin (A - 90^\circ)$  is positive; that is

$$\sin (A - 90^\circ) = .8.$$

$$\operatorname{cosec} (270^\circ - A) = \operatorname{cosec} (180^\circ + 90^\circ - A)$$

$$= -\operatorname{cosec} (90^\circ - A)$$

$$= -\left(\frac{1}{\pm .8}\right)$$

$$= \pm 1.25;$$

but between the given limits  $\operatorname{cosec} (270^\circ - A)$  must be positive, that is, the required value is 1.25.

116. By Art. 168,

$$\log_b c = \frac{\log_a c}{\log_a b}, \quad \log_c d = \frac{\log_a d}{\log_a c};$$

$$\text{hence the expression on the right} = \log_a b \times \frac{\log_a c}{\log_a b} \times \frac{\log_a d}{\log_a c}$$

$$= \log_a d.$$

$$\log_{10} 2 = 1 - \log_{10} 5 = \cdot 30103;$$

$$\log_{10} 8 = 3 \log_{10} 2 = \cdot 90309.$$

$$\log_8 10 = \frac{1}{\log_{10} 8} = 1 \cdot 1073093.$$

$$\begin{aligned} \log_{10} (\cdot 032)^5 &= 5 \log \frac{32}{1000} = 25 \log 2 - 15 \\ &= 7 \cdot 52575 - 15 \\ &= \bar{8} \cdot 52575. \end{aligned}$$

$$\begin{aligned} 117. \text{ First side} &= \cos (360^\circ + 60^\circ + A) + \cos (60^\circ - A) \\ &= \cos (60^\circ + A) + \cos (60^\circ - A) \\ &= 2 \cos 60^\circ \cos A = \cos A. \end{aligned}$$

For the second part, put  $A = 45^\circ$ .

118. Write  $t$  for  $\tan \frac{x}{2}$ , then the equation may be written

$$\frac{1-t^2}{1+t^2} - \sin \alpha \cot \beta \frac{2t}{1+t^2} = \cos \alpha;$$

$$\therefore t^2(1+\cos \alpha) + 2t \sin \alpha \cot \beta - (1-\cos \alpha) = 0;$$

$$t^2 + 2 \cot \beta \frac{\sin \alpha}{1+\cos \alpha} \cdot t - \frac{1-\cos \alpha}{1+\cos \alpha} = 0,$$

$$t^2 + 2 \cot \beta \tan \frac{\alpha}{2} t - \tan^2 \frac{\alpha}{2} = 0,$$

$$\left(t + \tan \frac{\alpha}{2} \cot \beta\right) \left(t - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}\right) = 0, \quad [\text{XI. d. Ex. 15.}]$$

$$\therefore \tan \frac{x}{2} = -\tan \frac{\alpha}{2} \cot \beta, \text{ or } \tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

119. We have

$$\frac{\sin (2A+B)}{\sin B} = \frac{m}{n};$$

$$\therefore \frac{m-n}{m+n} = \frac{\sin (2A+B) - \sin B}{\sin (2A+B) + \sin B}$$

$$= \frac{2 \cos (A+B) \sin A}{2 \sin (A+B) \cos A}$$

$$= \cot (A+B) \tan A.$$

120. Let  $x$  feet be the height of the tower;  
 then  $\angle ABD = 90^\circ - \angle ADB$   
 $= 45^\circ = \angle ADB$ ;  
 $\therefore AD = AB = x$ ;  
 $\therefore AC = x - 17$ .

Now from  $\triangle ABC$ ,

$$x^2 + (x - 17)^2 = 53^2;$$

$$\therefore x^2 - 17x - 1260 = 0,$$

$$(x - 45)(x + 28) = 0;$$

$$\therefore x = 45.$$

Again,

$$\tan ACB = \frac{45}{28};$$

but

$$\tan 31^\circ 48' = .62 = \frac{56}{90} = \frac{28}{45};$$

$$\therefore \angle ACB = 90^\circ - 31^\circ 48'$$

$$= 58^\circ 12'.$$

122.

$$\log 6 = \frac{1}{2} \log 36 = .778151,$$

$$\log 8 = \log 48 - \log 6 = 1.681241 - .778151$$

$$= .90309;$$

$$\therefore \log 2 = \frac{1}{3} \log 8 = .30103;$$

$$\therefore \log 3 = \log 6 - \log 2 = .477121.$$

Now

$$\log 40 = 1 + 2 \log 2 = 1.60206.$$

$$\log \sqrt{\frac{2}{15}} = \frac{1}{2} (\log 4 - \log 30)$$

$$= \frac{1}{2} (2 \log 2 - \log 3 - 1)$$

$$= \bar{1}.562469, \text{ on substitution.}$$

123. We have  $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{9}{10} \cot 72^\circ$ .

$$\log \tan \frac{B-C}{2} = 2 \log 3 - 1 + \log \cot 72^\circ$$

$$= \bar{1}.4660186$$

$$\log \tan 16^\circ 18' = \bar{1}.4660078$$

$$\frac{108}{4687} \times 60'' = 1''.$$

$$\therefore \frac{B-C}{2} = 16^\circ 18' 1'', \quad \frac{B+C}{2} = 18^\circ;$$

$$\therefore B = 34^\circ 18' 1'', \quad C = 1^\circ 41' 59''.$$

$$124. (1) \quad \cos A + \cos B \cos C = -\cos B + \overline{C} + \cos B \cos C \\ = \sin B \sin C; \\ \therefore \text{Second side} = a^2 \sin B \sin C = b \sin A \cdot c \sin A \\ = bc \sin^2 A.$$

$$(2) \quad \text{First side} = c(b \cos A + a \cos B) + 2ab \cos C \\ = c^2 + 2ab \cos C = a^2 + b^2.$$

$$125. \quad \tan \frac{\beta - \alpha}{2} = \frac{\tan \frac{\beta}{2} - \tan \frac{\alpha}{2}}{1 + \tan \frac{\beta}{2} \tan \frac{\alpha}{2}} = \frac{3 \tan \frac{\alpha}{2}}{1 + 4 \tan^2 \frac{\alpha}{2}} \\ = \frac{3 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + 4 \sin^2 \frac{\alpha}{2}} = \frac{3 \sin \alpha}{2 \cos^2 \frac{\alpha}{2} + 8 \sin^2 \frac{\alpha}{2}} \\ = \frac{3 \sin \alpha}{1 + \cos \alpha + 4(1 - \cos \alpha)} = \frac{3 \sin \alpha}{5 - 3 \cos \alpha}.$$

$$126. \quad \sin(36^\circ + A) - \sin(36^\circ - A) = 2 \cos 36^\circ \sin A = \frac{\sqrt{5} + 1}{2} \sin A; \\ \sin(72^\circ - A) - \sin(72^\circ + A) = 2 \cos 72^\circ \sin(-A) = -\frac{\sqrt{5} - 1}{2} \sin A.$$

By addition we obtain the required result.

$$127. \quad \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\frac{2}{3}}{\pm \sqrt{\frac{5}{9}}} = \pm \frac{2}{\sqrt{5}}.$$

The boundary line of  $\theta$  is in the 3rd or 4th quadrant, hence the tan is positive in one case and negative in the other.

$$128. (1) \quad \text{First side} = \sin 2A + \sin \left( \frac{\pi}{2} - 2B \right) \\ = 2 \sin \left( \frac{\pi}{4} + A - B \right) \cos \left( \frac{\pi}{4} - A - B \right).$$

$$(2) \quad \text{First side} = 2 \sin \frac{\theta - \phi}{2} \cos \frac{\theta + \phi}{2} \cdot 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\ = 2 \sin(\theta - \phi) \cos^2 \frac{\theta + \phi}{2}.$$

129. Since  $a : b : c = \sin A : \sin B : \sin C$ , we have

$$(a+b+c)(a+b-c) = 3ab,$$

or

$$(a+b)^2 - c^2 = 3ab,$$

that is,

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}; \therefore \cos C = \frac{1}{2}, \text{ and } C = 60^\circ.$$

130. Let  $a^x = b$ ,  $a^y = d$ ,  $c^p = d$ ,  $c^q = b$ ,

then we have to prove  $px = qy$ .

Now

$$a^x = c^q, \text{ and } a^y = c^p;$$

$$\therefore a^{px} = c^{pq} = a^{qy}; \text{ that is } px = qy.$$

131.

$$\log 20.01 = 1.3012471$$

$$\log 20.00 = 1.3010300$$

$$\text{diff. for } .01 = \frac{2171}{2171}$$

$$\therefore \text{diff. for } .0075 = \frac{3}{4} \times 2171 = 1628;$$

$$\therefore \log 20.0075 = 1.3011928.$$

132. Let  $AD$  be the median from  $B$ ; then

$$AB^2 + BC^2 = 2(AD^2 + BD^2);$$

that is,

$$49 + 81 = 2x^2 + 32;$$

whence

$$x = 7.$$

133. We have

$$\frac{1 + \sin A}{\cos A} = 2,$$

$$\therefore (1 + \sin A)^2 = 4(1 - \sin^2 A),$$

rejecting the factor  $1 + \sin A$ , which gives an inadmissible value, we have

$$1 + \sin A = 4(1 - \sin A);$$

whence

$$\sin A = \frac{3}{5}.$$

$$134. \text{ First side} = \frac{4(1 - \cos 2A) - (1 - \cos 4A)}{4(1 + \cos 2A) - (1 - \cos 4A)}$$

$$= \frac{8 \sin^2 A - 2 \sin^2 2A}{8 \cos^2 A - 2 \sin^2 2A}$$

$$= \frac{\sin^2 A (1 - \cos^2 A)}{\cos^2 A (1 - \sin^2 A)} = \tan^4 A.$$

$$\begin{aligned}
 135. \quad \text{First side} &= \frac{3 \sin A - 4 \sin^3 A + 4 \cos^3 A - 3 \cos A}{3 \sin A - 4 \sin^3 A - (4 \cos^3 A - 3 \cos A)} \\
 &= \frac{3(\sin A - \cos A) - 4(\sin^3 A - \cos^3 A)}{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)} \\
 &= \frac{\sin A - \cos A}{\sin A + \cos A} \cdot \frac{3 - 4(\sin^2 A + \cos^2 A + \sin A \cos A)}{3 - 4(\sin^2 A + \cos^2 A - \sin A \cos A)} \\
 &= \frac{\tan A - 1}{\tan A + 1} \cdot \frac{3 - 4(1 + \sin A \cos A)}{3 - 4(1 - \sin A \cos A)} \\
 &= \frac{\tan A - 1}{1 + \tan A} \cdot \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} = \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} \tan(A - 45^\circ).
 \end{aligned}$$

$$136. \quad \text{Put } \frac{x}{\cos A} = \frac{y}{\cos B} = k; \text{ then}$$

$$\begin{aligned}
 \text{First side} &= k \cos A \tan A + k \cos B \tan B \\
 &= k(\sin A + \sin B) \\
 &= 2k \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\
 &= 2k \cos \frac{A+B}{2} \cos \frac{A-B}{2} \tan \frac{A+B}{2} \\
 &= k(\cos A + \cos B) \tan \frac{A+B}{2} \\
 &= (x+y) \tan \frac{A+B}{2}.
 \end{aligned}$$

137.

$$\begin{aligned}
 \log 7 &= \log 24.5 - \log 3.5 = .845098; \\
 \log 5 &= \log 35 - \log 7 = .69897; \\
 \log 13 &= \log 3.25 - \log .25 \\
 &= \log 3.25 - (2 \log 5 - 2) = 1.113943.
 \end{aligned}$$

$$138. \quad \text{Here } \tan A = \frac{a}{b} = \frac{384}{330} = \frac{128}{110}.$$

$$\begin{array}{rcl}
 \log 128 & = & 7 \log 2 = 2.1072100 \\
 \log 110 & = & 2.0413927 \\
 \log \tan A & = & .0658173 \\
 \log \tan 49^\circ 19' & = & .0656886 \\
 \text{diff.} & & 1287
 \end{array}$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{1287}{2555} \times 60'' = 30'';$$

$$\therefore A = 49^\circ 19' 30''; \quad B = 40^\circ 40' 30''.$$

$$\begin{aligned}
 139. \text{ First side} &= \frac{2 \sin \frac{\theta + \alpha}{2} \sin \frac{\theta - \alpha}{2}}{2 \cos \frac{\theta + \alpha}{2} \cos \frac{\theta - \alpha}{2}} = \frac{\cos \alpha - \cos \theta}{\cos \alpha + \cos \theta} \\
 &= \frac{\cos \alpha - \cos \alpha \cos \beta}{\cos \alpha + \cos \alpha \cos \beta} = \frac{1 - \cos \beta}{1 + \cos \beta} \\
 &= \tan^2 \frac{\beta}{2}.
 \end{aligned}$$

140. We have

$$\begin{aligned}
 \frac{\sin \theta}{\cos \theta - \sin \phi} - \frac{\sin \theta}{\cos \theta + \sin \phi} &= \frac{2 \sin \theta \sin \phi}{\cos^2 \theta - \sin^2 \phi} \\
 &= \frac{\sin \phi \{ \cos \phi + \sin \theta - (\cos \phi - \sin \theta) \}}{1 - \sin^2 \theta - (1 - \cos^2 \phi)} \\
 &= \frac{\sin \phi \{ \cos \phi + \sin \theta - (\cos \phi - \sin \theta) \}}{(\cos \phi + \sin \theta) (\cos \phi - \sin \theta)} \\
 &= \frac{\sin \phi}{\cos \phi - \sin \theta} - \frac{\sin \phi}{\cos \phi + \sin \theta}.
 \end{aligned}$$

$$141. \text{ We have } \frac{c^2 (a+b)^2 s (s-b)}{ac} = \frac{b^2 (a+c)^2 s (s-c)}{ab};$$

$$c (a^2 + b^2 + 2ab) (s-b) = b (a^2 + c^2 + 2ac) (s-c);$$

$$\text{that is, } a^2 s (c-b) - bcs (c-b) + bc (c^2 - b^2) + 2abc (c-b) = 0,$$

$$\text{or } (c-b) \{ a^2 s - bc (s-c-b) + 2abc \} = 0,$$

$$\text{or } (c-b) \{ a^2 s - bc (a-s) + 2abc \} = 0,$$

$$\text{or } (c-b) \{ a^2 s + bcs + 3abc \} = 0;$$

therefore  $b-c=0$ , since the other factor evidently cannot be zero.

$$142. (1) \cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2}; \quad [\text{XI. d. Ex. 11}]$$

$$\begin{aligned}
 \tan A + \sec A &= \frac{1 + \sin A}{\cos A} = \frac{1 - \cos \left( \frac{\pi}{2} + A \right)}{\sin \left( \frac{\pi}{2} + A \right)} \\
 &= \tan \left( \frac{\pi}{4} + \frac{A}{2} \right);
 \end{aligned}$$

therefore by division the required result is obtained.

(2) Since  $\sin 3A = 3 \sin A - 4 \sin^3 A$ , we have

$$\begin{aligned} \text{First side} &= \frac{1}{4} \{ (3 \sin A - \sin 3A) + \dots + \dots \} \\ &= \frac{3}{4} \{ \sin A + \sin (120^\circ + A) + \sin (240^\circ + A) \} \\ &\quad - \frac{1}{4} \{ \sin 3A + \sin (360^\circ + 3A) + \sin (720^\circ + 3A) \} \\ &= -\frac{3}{4} \sin 3A. \quad [\text{See solution of XII. c.}] \end{aligned}$$

143. Let

then

$$\begin{aligned} x &= \sqrt[5]{18 \times .0015}, \\ x &= \sqrt[5]{.027} = (.3)^{\frac{3}{5}}, \\ \frac{3}{5} \log .3 &= \frac{3}{5} (1.4771213) \\ &= 1.6862728 \\ \log .48559 &= \frac{1.6862697}{31} \end{aligned}$$

diff. for .00001 =

$$\begin{aligned} \therefore \text{prop}^l \text{ increase} &= \frac{31}{90} \times .00001 \\ &= .0000344 \end{aligned}$$

$$\therefore x = .485593.$$

144.

$$\sin B = \frac{b \sin A}{a} = \frac{394}{573} \cos 22^\circ 4'.$$

$$\log 394 = 2.5954962$$

$$\log \cos 22^\circ 4' = 1.9669614$$

$$2.5624576$$

$$\log 573 = 2.7581546$$

$$\log \sin B = 1.8043030$$

$$\log \sin 39^\circ 35' = 1.8042757$$

$$\text{diff.} = 273$$

$$\text{diff. for } 60'' = 1527;$$

$$\therefore \text{prop}^l \text{ increase} = \frac{273}{1527} \times 60'' = 10.7'';$$

$$\therefore B = 39^\circ 35' 11''; \text{ and } C = 28^\circ 20' 49''.$$

$$145. \text{ First side} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc}.$$

$$146. \quad \log 119 = \log 7 + \log 17 = 2.0755469,$$

$$\log \frac{17}{7} = \log 17 - \log 7 = .3853509,$$

$$\begin{aligned} \log \frac{289}{343} &= \log 17^2 - \log 7^3 \\ &= 2 \log 17 - 3 \log 7 = 1.9256038. \end{aligned}$$

$$147. \quad \text{We have} \quad \cos \theta = \frac{\sin A}{\sin B + \sin C}$$

$$\begin{aligned} &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}} \\ &= \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}} = \frac{\cos \frac{B+C}{2}}{\cos \frac{B-C}{2}}; \end{aligned}$$

$$\therefore \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\cos \frac{B-C}{2} - \cos \frac{B+C}{2}}{\cos \frac{B-C}{2} + \cos \frac{B+C}{2}}$$

$$= \frac{2 \sin \frac{B}{2} \sin \frac{C}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

148. Let  $r$  be the radius of the circle,  $x, y$  the side of circumscribing equilateral triangle and hexagon respectively.

Then from the figure of Art. 215,

$$x = 2r \tan 60^\circ = 2r \sqrt{3}; \quad y = 2r \tan 30^\circ = \frac{2r}{\sqrt{3}};$$

whence

$$xy = 4r^2 = (2r)^2.$$

149. From the equation  $a^2 = b^2 + c^2 - 2bc \cos A$ , we have on substitution and reduction

$$c^2 - 150\sqrt{2} \cdot c + 10000 = 0;$$

$$\therefore (c - 100\sqrt{2})(c - 50\sqrt{2}) = 0;$$

$$\therefore c = 100\sqrt{2}, \text{ or } 50\sqrt{2}.$$

Again

$$\sin B = \frac{b \sin A}{a} = \frac{150}{50\sqrt{10}} = \frac{3}{\sqrt{10}},$$

$$\log \sin B = \log 3 - \frac{1}{2} \log 10$$

$$= 1.9771213,$$

$$\log \sin 71^\circ 33' = 1.9770832$$

diff.

$$381$$

$$\text{diff. for } 60'' = 421;$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{381}{421} \times 60'' = 54'';$$

$$\therefore B = 71^\circ 33' 54'', \text{ or } 180^\circ - (71^\circ 33' 54'').$$

150. The series may be written

$$(\operatorname{cosec} x - \operatorname{cosec} 3x) + (\operatorname{cosec} 3x - \operatorname{cosec} 3^2 x) + \dots + (\operatorname{cosec} 3^{n-1} x - \operatorname{cosec} 3^n x)$$

which reduces to

$$\operatorname{cosec} x - \operatorname{cosec} 3^n x.$$

153. First side

$$= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} - \frac{3 \cos \frac{3\theta}{2}}{\sin \frac{3\theta}{2}}$$

$$= \frac{\sin \frac{3\theta}{2} \cos \frac{\theta}{2} - 3 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{\sin \frac{\theta}{2} \sin \frac{3\theta}{2}}$$

$$= \frac{\sin \left( \frac{3\theta}{2} - \frac{\theta}{2} \right) - 2 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{\sin \frac{\theta}{2} \sin \frac{3\theta}{2}}$$

$$= \frac{\sin \theta - (\sin 2\theta - \sin \theta)}{\frac{1}{2} (\cos \theta - \cos 2\theta)}$$

$$= \frac{4 \sin \theta - 2 \sin 2\theta}{1 + \cos \theta - 2 \cos^2 \theta} = \frac{4 \sin \theta (1 - \cos \theta)}{(1 + 2 \cos \theta) (1 - \cos \theta)}$$

$$= \frac{4 \sin \theta}{1 + 2 \cos \theta}$$

154. We have, by Art. 135, Ex. 2,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C:$$

$$\therefore \frac{3}{4} + \frac{5}{12} + \tan C = \frac{3}{4} \cdot \frac{5}{12} \cdot \tan C;$$

whence

$$\tan C = -\frac{56}{33}.$$

Also

$$\cos A = \frac{1}{\sqrt{1 + \tan^2 A}} = \frac{1}{\sqrt{1 + \frac{9}{16}}} = \frac{4}{5},$$

$$\cos B = \frac{1}{\sqrt{1 + \tan^2 B}} = \frac{1}{\sqrt{1 + \frac{25}{144}}} = \frac{12}{13},$$

$$\cos C = -\frac{1}{\sqrt{1 + \tan^2 C}} = -\frac{1}{\sqrt{1 + \frac{3136}{1089}}} = -\frac{33}{65},$$

the negative sign of the radical being taken in the third case since  $C$  is an obtuse angle.

Again  $\tan C = -\frac{56}{33}; \therefore \tan (180^\circ - C) = \frac{56}{33},$

$$\log 56 = 1.7481880$$

$$\log 33 = 1.5185139$$

$$\log \tan (180^\circ - C) = .2296741$$

$$\log \tan 59^\circ 29' = .2295627$$

$$\text{diff.} \quad \quad \quad 1114$$

$$\therefore \text{prop}^l \text{ increase} = \frac{1114}{2888} \times 60' = 23'';$$

$$\therefore 180^\circ - C = 59^\circ 29' 23'';$$

that is,

$$C = 120^\circ 30' 37''.$$

155. We have

$$\frac{\sin (A - B)}{\sin (A + B)} = \frac{a^2 - b^2}{a^2 + b^2}$$

$$= \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B};$$

$$\therefore \frac{\sin (A - B)}{\sin C} = \frac{\sin (A - B) \sin (A + B)}{\sin^2 A + \sin^2 B};$$

$\therefore$  either

$$\sin (A - B) = 0 \dots\dots\dots (1),$$

or

$$\sin^2 C = \sin^2 A + \sin^2 B \dots\dots\dots (2).$$

If (1) is true,  $A = B$ ; if (2) is true, we have  $c^2 = a^2 + b^2$ .

$$\begin{aligned}
 156. \quad \text{The first side} &= \frac{2\Delta}{s} \cdot \frac{abc}{\Delta} \left\{ \frac{s(s-a)}{bc} + \dots + \dots \right\} \\
 &= 2 \{ a(s-a) + b(s-b) + c(s-c) \} \\
 &= (a+b+c)^2 - 2a^2 - 2b^2 - 2c^2 \\
 &= 2bc + 2ca + 2ab - a^2 - b^2 - c^2.
 \end{aligned}$$

157. We have

$$\begin{aligned}
 \frac{\cos C}{\sin B \cos A} - \frac{\cos B}{\sin C \cos A} &= \frac{\cos(A+C)}{\sin C \cos A} - \frac{\cos(A+B)}{\sin B \cos A} \\
 &= \cot C - \tan A - (\cot B - \tan A) \\
 &= \cot C - \cot B.
 \end{aligned}$$

$$\begin{aligned}
 158. \quad \log 3 &= \log 18 - \log 6; \quad \log 2 = \log 6 - \log 3, \\
 \log 11 &= \log 44 - \log 4 = \log 44 - 2 \log 2.
 \end{aligned}$$

159. (1) We have

$$\begin{aligned}
 &\tan(60^\circ + A) \tan(60^\circ - A) \\
 &= \frac{2 \sin(60^\circ + A) \sin(60^\circ - A)}{2 \cos(60^\circ + A) \cos(60^\circ - A)} = \frac{\cos 2A - \cos 120^\circ}{\cos 2A + \cos 120^\circ} = \frac{2 \cos 2A + 1}{2 \cos 2A - 1} \\
 \therefore \text{second side} &= 2 \sin A \cos 2A + \sin A \\
 &= (\sin 3A - \sin A) + \sin A = \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{First side} &= 2 \sin(A-B) \cos(A+B) \frac{\sin(A+B)}{\cos(A+B)} \\
 &= 2 \sin(A+B) \sin(A-B) \\
 &= 2 \sin^2 A - \sin^2 B. \quad [\text{Art. 114.}]
 \end{aligned}$$

$$160. \quad \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{113 \times 101}{296 \times 82}}$$

$$\begin{aligned}
 \log 296 &= 2.4712917 \\
 \log 82 &= 1.9138139 \\
 \hline
 &4.3851056
 \end{aligned}$$

$$\begin{aligned}
 \log 113 &= 2.0530784 \\
 \log 101 &= 2.0043214
 \end{aligned}$$

$$\begin{aligned}
 &4.0573998 \\
 &4.3851056 \\
 \hline
 &2 \overline{1.6722942}
 \end{aligned}$$

$$\begin{aligned}
 \log \tan \frac{C}{2} &= 1.8361471 \\
 \log \tan 34^\circ 26' &= 1.8360513 \\
 \hline
 \text{diff.} &958
 \end{aligned}$$

$$\text{prop}^l. \text{ increase} = \frac{958}{2708} \times 60'' = 21'';$$

$$\therefore C = 68^\circ 52' 42''.$$

161. Let  $a$  be a side of the octagon,  $r$  the radius of the circle, then  $8a = 2\pi r$ ; and we have

$$\begin{aligned}\frac{\text{area of circle}}{\text{area of octagon}} &= \frac{\pi r^2}{2a^2 \cot \frac{\pi}{8}} = \frac{8\pi r^2}{\pi^2 r^2} \tan \frac{\pi}{8} \\ &= \frac{8(\sqrt{2}-1)}{\pi} = \frac{.414 \times 8}{3.1416} = \frac{4140}{3927} = \frac{1380}{1309}.\end{aligned}$$

162. We have  $2b = a + c$ , or  $a = 2b - c$ ;

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b - c)^2}{2bc} = \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c}.$$

163. We have  $a(\sin \theta \cos \alpha + \cos \theta \sin \alpha) = b(\sin \theta \cos \beta + \cos \theta \sin \beta)$ ;

$$\therefore \sin \theta (a \cos \alpha - b \cos \beta) = \cos \theta (b \sin \beta - a \sin \alpha);$$

that is, 
$$\cot \theta = \frac{a \cos \alpha - b \cos \beta}{b \sin \beta - a \sin \alpha}.$$

164. Let  $a, b, A$  be the given parts; then  $R = \frac{a}{2 \sin A}$ , which is the same for each triangle.

165. Let  $NS$  be a horizontal line pointing North and South. Then if  $K$  be the position of the kite, and  $KD$  is the vertical from  $K$ , we have  $S, B, D, A, N$  in a straight line, and  $BA = c$ . Also  $KA = \frac{c \sin \beta}{\sin BKA} = \frac{c \sin \beta}{\sin (\alpha + \beta)}$ .

And  $KD = KA \sin \alpha = \frac{c \sin \alpha \sin \beta}{\sin (\alpha + \beta)}.$

$$\begin{aligned}166. \quad \cos \alpha + \cos \beta + \cos \gamma + 1 &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\gamma}{2} \\ &= 2 \cos \left( \pi - \frac{\gamma}{2} \right) \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\gamma}{2} \\ &= -2 \cos \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\gamma}{2} \\ &= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\gamma}{2} - \cos \frac{\alpha - \beta}{2} \right\} \\ &= 2 \cos \frac{\gamma}{2} \left\{ \cos \left( \pi - \frac{\alpha + \beta}{2} \right) - \cos \frac{\alpha - \beta}{2} \right\} \\ &= -2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha + \beta}{2} + \cos \frac{\alpha - \beta}{2} \right\} \\ &= -4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.\end{aligned}$$

$$\begin{aligned}
 167. \quad \text{First side} &= 2 \cos 9A \cos A + 6 \cos 3A \cos A \\
 &= 2 \cos A \{ \cos 9A + 3 \cos 3A \} \\
 &= 2 \cos A \{ (4 \cos^3 3A - 3 \cos 3A) + 3 \cos 3A \} \\
 &= 8 \cos A \cos^3 3A.
 \end{aligned}$$

$$168. \quad r_1 + r_2 = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = \frac{\Delta (s-a+s-b)}{(s-a)(s-b)} = c \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

$$\text{Also} \quad r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2 \quad [\text{XVIII. a. Ex. 24}]$$

$$\begin{aligned}
 \therefore \text{First side} &= \frac{abc}{4s^2} \sqrt{\frac{s(s-c)}{(s-a)(s-b)} \cdot \frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)}} \\
 &= \frac{abc}{4 \sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\Delta} = R.
 \end{aligned}$$

$$\begin{aligned}
 169. \quad \cos 2\theta &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = -\frac{2 \tan^2 \phi}{2(1 + \tan^2 \phi)} \\
 &= -\sin^2 \phi.
 \end{aligned}$$

$$\begin{aligned}
 170. \quad \text{First side} &= \frac{\sin A \sin (60^\circ + A) \sin (120^\circ + A)}{\cos A \cos (60^\circ + A) \cos (120^\circ + A)} \\
 &= \frac{\sin A}{\cos A} \cdot \frac{\cos 60^\circ - \cos (180^\circ + 2A)}{\cos 60^\circ + \cos (180^\circ + 2A)} \\
 &= \frac{\sin A}{\cos A} \cdot \frac{1 + 2 \cos 2A}{1 - 2 \cos 2A} \\
 &= \frac{\sin A + 2 \sin A \cos 2A}{\cos A - 2 \cos A \cos 2A} \\
 &= \frac{\sin A + (\sin 3A - \sin A)}{\cos A - (\cos 3A + \cos A)} \\
 &= -\frac{\sin 3A}{\cos 3A} = -\tan 3A.
 \end{aligned}$$

$$171. \quad \text{We have } A - B = B; \text{ therefore } \sin (A - B) = \sin B;$$

$$\begin{aligned}
 \text{also} \quad \sin (A + B) &= \sin (180^\circ - C) = \sin C; \\
 \therefore \sin (A + B) \sin (A - B) &= \sin C \sin B;
 \end{aligned}$$

$$\text{that is,} \quad \sin^2 A - \sin^2 B = \sin B \sin C;$$

$$\text{or} \quad a^2 - b^2 = bc.$$

172. If  $a, b$  be the sides of the triangle and square respectively, and  $R$  the radius of the circle, it is easy to shew that

$$a = R\sqrt{3}; \quad b = 2R \cos 45^\circ = R\sqrt{2}.$$

$$\begin{aligned} 173. \quad \text{We have } \frac{c+b}{c-b} \tan \frac{A}{2} &= \frac{\sin C + \sin B}{\sin C - \sin B} \tan \frac{A}{2} \\ &= \frac{\sin(A+B) + \sin B}{\sin(A+B) - \sin B} \tan \frac{A}{2} \\ &= \frac{2 \sin\left(\frac{A}{2} + B\right) \cos \frac{A}{2}}{2 \cos\left(\frac{A}{2} + B\right) \sin \frac{A}{2}} \tan \frac{A}{2} \\ &= \tan\left(\frac{A}{2} + B\right). \end{aligned}$$

$$\tan\left(\frac{A}{2} + B\right) = \frac{7b+3b}{7b-3b} \tan \frac{A}{2} = \frac{10}{4} \tan \frac{A}{2},$$

$$\log 10 = 1$$

$$\log 4 = .60206$$

$$.3979400$$

$$\log \tan 3^\circ 18' 42'' = \bar{2}.7624069$$

$$\log \tan\left(\frac{A}{2} + B\right) = \bar{1}.1603469$$

$$\log \tan 8^\circ 13' 50'' = \bar{1}.1603083$$

$$386$$

$$\begin{aligned} \text{prop}^l \text{ increase} &= \frac{386}{1486} \times 60' \\ &= 2.6''. \end{aligned}$$

$$\therefore \frac{A}{2} + B = 8^\circ 13' 53'', \text{ and } \frac{A}{2} = 3^\circ 18' 42'';$$

$$\therefore B = 4^\circ 55' 11'',$$

$$C = 168^\circ 27' 25''.$$

174. By a well-known geometrical property, we have

$$AC^2 + AB^2 = 2AD^2 + 2DB^2.$$

$$\therefore AC^2 - AB^2 = 2(AD^2 + DB^2 - AB^2)$$

$$= 4AD \cdot DB \cos ADB$$

$$= 4AD \cdot DB \sin ADB \cot ADB$$

$$= (AD \cdot DB \sin ADB) 4 \cot ADB$$

$$= 4\Delta \cot ADB,$$

for  $AD \cdot DB \sin ADB = 2$  (area of triangle  $ADB$ )  $= \Delta$ .

$$\begin{aligned}
 175. \quad \tan 40^\circ \tan 80^\circ &= \frac{2 \sin 40^\circ \sin 80^\circ}{2 \cos 40^\circ \cos 80^\circ} \\
 &= \frac{\cos 40^\circ - \cos 120^\circ}{\cos 40^\circ + \cos 120^\circ} \\
 &= \frac{2 \cos 40^\circ + 1}{2 \cos 40^\circ - 1};
 \end{aligned}$$

$$\begin{aligned}
 \therefore \tan 20^\circ \tan 40^\circ \tan 80^\circ &= \frac{2 \sin 20^\circ \cos 40^\circ + \sin 20^\circ}{2 \cos 20^\circ \cos 40^\circ - \cos 20^\circ} \\
 &= \frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ.
 \end{aligned}$$

176. See Art. 144.

$$177. \quad 2R = \frac{abc}{2\Delta}; \quad 2r = \frac{2\Delta}{s};$$

$$\therefore 2R \cdot 2r = \frac{abc}{s} = \frac{2abc}{a+b+c}.$$

$$178. \quad \sin B = \frac{b}{a} \sin A = \frac{4\sqrt{3}}{7}$$

$$\frac{1}{2} \log 3 = .2385606$$

$$2 \log 2 = .6020600$$

$$.8406206$$

$$\log 7 = .8450980$$

$$\log \sin B = \bar{1}.9955226$$

$$\log \sin 81^\circ 47' = \bar{1}.9955188$$

$$38$$

$$\begin{aligned}
 \text{prop}^l. \text{ increase} &= \frac{38}{183} \times 60'' \\
 &= 12.4''.
 \end{aligned}$$

$\therefore B = 81^\circ 47' 12''$ ; but since  $a < b$  there is another value of  $B$  supplementary to this, viz.  $98^\circ 12' 48''$ .

$$\therefore C = 68^\circ 12' 48'', \text{ or } 51^\circ 47' 12''.$$

$$\text{To find } c, \text{ we have } c^2 - 2b \cos A \cdot c + b^2 - a^2 = 0$$

[Art. 150]

that is

$$c^2 - 24c + 143 = 0,$$

$$(c - 13)(c - 11) = 0;$$

$$\therefore c = 13, \text{ or } 11.$$

$$179. \quad \tan^2 \left( \frac{\pi}{4} + \beta \right) = \frac{1 + \sin 2\beta}{1 - \sin 2\beta} \quad [\text{XI. f. Ex. 15}]$$

$$= \frac{1 + \sin 2a \sin 2a' + \sin 2a + \sin 2a'}{1 + \sin 2a \sin 2a' - \sin 2a - \sin 2a'}$$

$$= \frac{(1 + \sin 2a)(1 + \sin 2a')}{(1 - \sin 2a)(1 - \sin 2a')}$$

$$= \tan^2 \left( \frac{\pi}{4} + a \right) \tan^2 \left( \frac{\pi}{4} + a' \right).$$

180. With the figure of Art. 199, let

$$PC = x, \beta = 28^\circ, a = 16^\circ, a = 16071 \text{ feet.}$$

Then

$$x = \frac{16071 \sin 28^\circ \sin 16^\circ}{\sin 12^\circ}.$$

$$\log 16071 = 4.2060$$

$$\log \sin 28^\circ = 1.6716$$

$$\log \sin 16^\circ = 1.4403$$

$$3.3179$$

$$\log \sin 12^\circ = 1.3179$$

$$\log x = 4.0000$$

$$\therefore x = 10000 \text{ feet.}$$

$$182. \quad \text{We have } 2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} = 2 \sin (A+C)$$

$$= 4 \sin \frac{A+C}{2} \cos \frac{A+C}{2};$$

$$\therefore \text{either} \quad \cos \frac{A+C}{2} = 0 \dots\dots\dots(1),$$

$$\text{or} \quad \cos \frac{A-C}{2} = 2 \sin \frac{A+C}{2} \dots\dots\dots(2).$$

From (1)  $\frac{A+C}{2} = (2n+1)\frac{\pi}{2}$ , and from (2) by expanding each side and dividing throughout by  $\cos \frac{A}{2} \cos \frac{C}{2}$  we obtain the other result.

$$183. \quad \text{First side} = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{C}{2} \right\}$$

$$\begin{aligned}
 &= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{\pi - (A+B)}{2} \right\} \\
 &= 2 \sin \frac{C}{2} \cdot 2 \sin \left( \frac{\pi}{4} - \frac{B}{2} \right) \sin \left( \frac{\pi}{4} - \frac{A}{2} \right) \\
 &= 4 \sin \frac{C}{2} \sin \left( 45^\circ - \frac{A}{2} \right) \sin \left( 45^\circ - \frac{B}{2} \right).
 \end{aligned}$$

$$184. \quad \frac{bc}{r_1} = \frac{bc(s-a)}{\Delta} = \frac{4R(s-a)}{a} = 2R \left( \frac{b+c-a}{a} \right);$$

$$\begin{aligned}
 \therefore \text{first side} &= 2R \left( \frac{b+c-a}{a} + \frac{c+a-b}{b} + \frac{a+b-c}{c} \right) \\
 &= 2R \left( \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3 \right).
 \end{aligned}$$

185. Since the points  $A, B, E, C$  are concyclic,  $\angle BED = \angle C$ ; also

$$\angle EBD = \angle EAC = \frac{A}{2};$$

$\therefore$  from  $\triangle BDE$ ,

$$BD = \frac{DE \sin C}{\sin \frac{A}{2}},$$

from  $\triangle DEC$ ,

$$DC = \frac{DE \sin B}{\sin \frac{A}{2}};$$

$$\therefore \text{by addition } a = \frac{DE}{\sin \frac{A}{2}} (\sin B + \sin C);$$

$$\therefore a^2 = \frac{DE}{\sin \frac{A}{2}} (a \sin B + a \sin C)$$

$$= \frac{DE (b+c) \sin A}{\sin \frac{A}{2}};$$

$$\therefore DE = \frac{a^2 \sec \frac{A}{2}}{2(b+c)}.$$

$$186. \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{5875.5 \times 1785.5}{3850 \times 3811}}$$

$$\log 5875.5 = 3.7690448$$

$$\log 1785.5 = 3.2517599$$

$$7.0208047$$

$$7.1664996$$

$$2 \overline{1.8543051}$$

$$\log \cos \frac{A}{2} = \overline{1.9271525}$$

$$\log \cos 32^\circ 16' = \overline{1.9271509}$$

$$16$$

$$\log 3850 = 3.5854607$$

$$\log 3811 = 3.5810389$$

$$7.1664996$$

$$\text{prop}^l. \text{ decrease} = \frac{16}{797} \times 60'' \\ = 1.2'';$$

$$\therefore \frac{A}{2} = 32^\circ 15' 58.8'', \text{ or } A = 64^\circ 31' 58''.$$

188. See Art. 117.

$$190. \quad (1) \quad \frac{b^2 + c^2 - a^2}{a^2 + c^2 - b^2} = \frac{2bc \cos A}{2ca \cos B} = \frac{b \cos A}{a \cos B} \\ = \frac{\sin B \cos A}{\sin A \cos B} = \frac{\tan B}{\tan A}.$$

$$(2) \quad \frac{2 \sin^2 A}{a^2} = \frac{2 \sin^2 B}{b^2};$$

$$\therefore \frac{1 - \cos 2A}{a^2} = \frac{1 - \cos 2B}{b^2};$$

that is,

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

$$191. \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{110.33.42.35} \\ = \sqrt{11^2.7^2.3^2.2^2.5^2} = 11.7.3.2.5 \\ = 2310 \text{ sq. ft.}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{2310}{33} = 70 \text{ ft.}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{2310}{42} = 55 \text{ ft.}$$

$$r_3 = \frac{\Delta}{s-c} = \frac{2310}{35} = 66 \text{ ft.}$$

192. Draw  $DK, DK'$  perpendicular to  $AB, AC$  respectively; then

$$DK \cdot AB + DK' \cdot AC = 2\Delta;$$

that is,

$$c \cdot AD \sin \frac{A}{2} + b \cdot AD \sin \frac{A}{2} = bc \sin A;$$

$$\therefore AD(b+c) = 2bc \cos \frac{A}{2}.$$

193. (1)

$$\sin 5\theta - \sin 3\theta = \sqrt{2} \sin \theta.$$

$$\therefore 2 \sin \theta \cos 4\theta = \sqrt{2} \sin \theta;$$

that is,

$$\sin \theta = 0, \text{ or } \cos 4\theta = \frac{1}{\sqrt{2}}.$$

$$\therefore \theta = n\pi, \text{ or } 4\theta = 2n\pi \pm \frac{\pi}{4}.$$

(2)

$$\cot \theta + \cot \left( \frac{\pi}{4} + \theta \right) = 2;$$

$$\therefore \cot \theta + \frac{\cot \theta - 1}{\cot \theta + 1} = 2;$$

$$\cot^2 \theta + 2 \cot \theta - 1 = 2 \cot \theta + 2;$$

$$\cot^2 \theta = 3;$$

that is,

$$\cot \theta = \pm \sqrt{3}, \text{ and } \theta = n\pi \pm \frac{\pi}{6}.$$

194. The given relation easily reduces to  $\cos 2\alpha = \sin 2\beta$ , one solution of which is  $2\alpha = \frac{\pi}{2} - 2\beta$ .

195. We have  $\tan(\alpha + \theta) \tan(\alpha - \theta) = \tan^2 \beta;$

$$\therefore \frac{\tan^2 \alpha - \tan^2 \theta}{1 - \tan^2 \alpha \tan^2 \theta} = \tan^2 \beta;$$

whence

$$\tan^2 \theta (1 - \tan^2 \alpha \tan^2 \beta) = \tan^2 \alpha - \tan^2 \beta;$$

$$\begin{aligned} \therefore \tan^2 \theta &= \frac{(\tan \alpha + \tan \beta)(\tan \alpha - \tan \beta)}{(1 - \tan \alpha \tan \beta)(1 + \tan \alpha \tan \beta)} \\ &= \tan(\alpha + \beta) \tan(\alpha - \beta). \end{aligned}$$

196. (1) We have

$$p_1 = \frac{2\Delta}{a};$$

$\therefore$  second side

$$= \frac{a^3 b^3 c^3}{8\Delta^3} = 8 \left( \frac{abc}{4\Delta} \right)^3 = 8R^3.$$

$$(2) \text{ Second side} = \frac{a^2 + b^2 - 2ab \cos C}{4\Delta^2}$$

$$= \frac{c^2}{4\Delta^2} = \left( \frac{c}{2\Delta} \right)^2 = \frac{1}{p_3^2}.$$

197. By Art. 215 we have

$$\text{perimeter} = 30r \tan \frac{\pi}{15}, \text{ where } \pi r^2 = 1386.$$

Now  $r^2 = \frac{7}{22} \times 1386 = 7 \times 63; \therefore r = 21;$

$$\begin{aligned} \therefore \text{perimeter} &= 30 \times 21 \tan 12^\circ \\ &= 630 \times .213 \\ &= 134.19 \text{ ft.} \end{aligned}$$

198. Let  $A, B$  represent the foot of the pole in the two positions;  $C, S$  the top of the pole on the coping and sill respectively; also let  $W$  be the foot of the wall.

Then  $x + SW = AC \sin \alpha,$   
 but  $SW = BS \sin \beta = AC \sin \beta;$   
 $\therefore x = AC (\sin \alpha - \sin \beta).$

Similarly  $a = AC (\cos \beta - \cos \alpha);$

$$\therefore \frac{x}{a} = \frac{2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}{2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}} = \cot \frac{\alpha + \beta}{2}.$$

199. See Examples XVIII. a. 18. Each of the three expressions will be found to be equal to  $r$ .

200. (1) Let  $\sin^{-1} \frac{2}{7} = \theta$ ; then  $\cos 2\theta = 1 - 2 \sin^2 \theta = \frac{41}{49};$

$$\therefore \cos^{-1} \frac{41}{49} = 2\theta = 2 \sin^{-1} \frac{2}{7}.$$

$$\begin{aligned} (2) \quad 3 \tan^{-1} \frac{1}{4} &= \tan^{-1} \frac{3 \times \frac{1}{4} - \left(\frac{1}{4}\right)^3}{1 - 3 \left(\frac{1}{4}\right)^2} \\ &= \tan^{-1} \frac{3 \times 16 - 1}{64 - 12} = \tan^{-1} \frac{47}{52}. \end{aligned}$$

201. (1) As in XI. f. Ex. 14 we may prove that

$$\tan A + \sec A = \tan \left( 45^\circ + \frac{A}{2} \right).$$

Also  $\cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2};$

$$\therefore (\tan A + \sec A) \cot \frac{A}{2} = (\cot A + \operatorname{cosec} A) \tan \left( 45^\circ + \frac{A}{2} \right).$$

(2) First side

$$= 2 \cos (A+B) \cos (A-B) - 2 \cos (A+B) \{ \cos (A-B) - \cos (90^\circ - A+B) \}$$

$$= 2 \cos (A+B) \sin (A+B) = \sin 2(A+B).$$

202. By Art. 214,  $\frac{\text{perimeter of fig.}}{\text{diameter of circle}} = 7 \sin 25\frac{1}{4}^\circ$ ,

$$\log 7 \sin 25\frac{1}{4}^\circ = .8450980 + \bar{1}.6373733$$

$$= .4824713, \text{ which is greater than } \log 3.$$

203. We have  $c^2 = a^2 + b^2 - 2ab \cos C$

$$= (a^2 + b^2) \left( \cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left( \cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right)$$

$$= (a+b)^2 \sin^2 \frac{C}{2} + (a-b)^2 \cos^2 \frac{C}{2}$$

$$= (a+b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left( \frac{a-b}{a+b} \right)^2 \cot^2 \frac{C}{2} \right\}$$

$$= (a+b)^2 \sin^2 \frac{C}{2} (1 + \tan^2 \phi)$$

$$= (a+b)^2 \sin^2 \frac{C}{2} \sec^2 \phi;$$

$$\therefore c = (a+b) \sin \frac{C}{2} \sec \phi.$$

204. We have  $\tan \phi = \frac{237-158}{237+158} \cot 33^\circ 20'$

$$= \frac{2}{10} \cot 33^\circ 20';$$

$$\therefore \log \tan \phi = \log 2 - 1 + \log \cot 33^\circ 20'$$

$$= \bar{1}.30103 + .18197$$

$$= \bar{1}.48300$$

$$\log \tan 16^\circ 54' = \bar{1}.48262$$

$$\text{diff.} \quad \quad \quad 38$$

$$\therefore \phi = 16^\circ 54' 50'',$$

$$\log \sec 16^\circ 55' = .01921$$

$$\log \sec 16^\circ 54' = .01917$$

$$\text{diff. for } 60'' \quad \quad \quad 4$$

$$\therefore \log \sec \phi = .01920.$$

$$\text{prop}^l \text{ increase} = \frac{38}{46} \times 60''$$

$$= 50'';$$

$$\text{prop}^l \text{ increase for } 50'' = \frac{4}{60} \times 50''$$

$$= 3'';$$

Now

$$c = (a + b) \sin \frac{C}{2} \sec \phi = 395 \sin \frac{C}{2} \sec \phi,$$

$$\begin{aligned} \log 395 &= \log 79 + 1 - \log 2 \\ &= 2.59660 \end{aligned}$$

$$\log \sin \frac{C}{2} = 1.73998$$

$$\log \sec \phi = .01920$$

$$\log c = 2.35578$$

$$\therefore c = 226.87.$$

205.

$$2 \cos^2 2\theta = 1 + \cos 4\theta;$$

$$\therefore 2 \cos 2\theta = \sqrt{2 + 2 \cos 4\theta}.$$

Similarly

$$2 \cos \theta = \sqrt{2 + 2 \cos 2\theta} = \sqrt{2 + \sqrt{2 + 2 \cos 4\theta}}.$$

206. Let

$$\sin^{-1} \frac{3}{\sqrt{73}} = \alpha, \text{ so that } \cos \alpha = \frac{8}{\sqrt{73}},$$

and let

$$\cos^{-1} \frac{11}{\sqrt{146}} = \beta, \text{ so that } \sin \beta = \frac{5}{\sqrt{146}}.$$

Then

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \frac{3}{\sqrt{73}} \cdot \frac{11}{\sqrt{146}} + \frac{8}{\sqrt{73}} \cdot \frac{5}{\sqrt{146}} = \frac{73}{73\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \sin \frac{\pi}{4} = \sin \left( \frac{5\pi}{12} - \frac{\pi}{6} \right);$$

$$\therefore \alpha + \beta = \frac{5\pi}{12} - \frac{\pi}{6} = \frac{5\pi}{12} - \sin^{-1} \frac{1}{2}.$$

Again

$$\tan^{-1} \frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \frac{x-1}{x+1} \cdot \frac{2x-1}{2x+1}} = \tan^{-1} \frac{23}{36};$$

$$\tan^{-1} \frac{4x^2 - 2}{6x} = \tan^{-1} \frac{23}{36};$$

$$\therefore 36(2x^2 - 1) = 69x,$$

or

$$24x^2 - 23x - 12 = 0;$$

$$\therefore (3x - 4)(8x + 3) = 0,$$

that is,

$$x = \frac{4}{3}, \text{ or } -\frac{3}{8}.$$

207. We have  $x = \frac{2\Delta}{a}, R = \frac{abc}{4\Delta};$

$$\therefore x = \frac{1}{a} \cdot \frac{abc}{2R}; \therefore \frac{bx}{c} = \frac{b^2}{2R},$$

that is,  $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2 + b^2 + c^2}{2R}.$

208. We have  $\cos(\alpha + \theta) = \cos \left\{ \frac{\pi}{2} - (\alpha - \theta) \right\},$

$$\therefore \alpha + \theta = 2m\pi \pm \left\{ \frac{\pi}{2} - (\alpha - \theta) \right\};$$

the upper sign gives  $2\alpha = 2m\pi + \frac{\pi}{2},$

and the lower sign gives  $2\theta = 2m\pi - \frac{\pi}{2}.$

209. With the notation of Art. 228,  
 $SI^2 = R^2 - 2Rr.$

If  $a$  be the base of the triangle,  $A = 120^\circ,$   
 $B = C = 30^\circ; \therefore r = 4R \sin 60^\circ \sin 15^\circ \sin 15^\circ.$

$$\therefore SI^2 = R^2 - 8R^2 \frac{\sqrt{3}}{2} \left( \frac{\sqrt{3}-1}{2\sqrt{2}} \right)^2$$

$$= R^2 (4 - 2\sqrt{3});$$

$$\therefore SI = R(\sqrt{3}-1)$$

$$= \frac{a}{2 \sin A} (\sqrt{3}-1) = \frac{a(\sqrt{3}-1)}{\sqrt{3}},$$

$$\therefore SI : a = \sqrt{3}-1 : \sqrt{3}.$$

210.  $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$$= 1 + \frac{r}{R} = 1 + \frac{3}{4},$$

$$\therefore 4(\cos A + \cos B + \cos C) = 7.$$

211. Since  $\frac{\cos(\theta - \alpha)}{\sin(\theta + \alpha)} = \frac{1+m}{1-m},$

dividendo and componendo, we have

$$\frac{\cos(\theta - \alpha) - \sin(\theta + \alpha)}{\cos(\theta - \alpha) + \sin(\theta + \alpha)} = m.$$

By expanding the sines and cosines we obtain

$$\frac{(\cos \theta - \sin \theta)(\cos a - \sin a)}{(\cos \theta + \sin \theta)(\cos a + \sin a)} = m,$$

or 
$$\frac{1 - \tan \theta}{1 + \tan \theta} = m \left( \frac{\cot a + 1}{\cot a - 1} \right). \quad [\text{See XI. b. Ex. 6, 7.}]$$

212. (1) 
$$\begin{aligned} \sin 5\theta - \sin 3\theta &= \sqrt{2} \cos 4\theta; \\ 2 \cos 4\theta \sin \theta &= \sqrt{2} \cos 4\theta; \\ \therefore \cos 4\theta &= 0; \text{ whence } 4\theta = (2n+1) \frac{\pi}{2}, \end{aligned}$$

or 
$$\sin \theta = \frac{1}{\sqrt{2}}; \text{ whence } \theta = n\pi + (-1)^n \frac{\pi}{4}.$$

(2) 
$$1 + \sin 2\theta = \frac{1 + \tan \theta}{1 - \tan \theta}.$$

$$1 + \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{1 + \tan \theta}{1 - \tan \theta}.$$

$$(1 + \tan \theta)^2 (1 - \tan \theta) = (1 + \tan \theta) (1 + \tan^2 \theta);$$

$$\therefore 1 + \tan \theta = 0; \text{ whence } \theta = n\pi + \frac{3\pi}{4},$$

or 
$$1 - \tan^2 \theta = 1 + \tan^2 \theta;$$

$$\therefore \tan \theta = 0; \text{ whence } \theta = n\pi.$$

213. We have 
$$2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} = 4 \sin^2 \frac{C}{2};$$

$$\therefore \cos \frac{A-B}{2} = 2 \sin \frac{C}{2}, \text{ or } 2 \cos \frac{A-B}{2} \sin \frac{A+B}{2} = 4 \sin \frac{C}{2} \cos \frac{C}{2};$$

that is, 
$$\sin A + \sin B = 2 \sin C, \text{ or } a + b = 2c.$$

214. With the figure on p. 186, we have  $\tan \beta = \frac{1}{9}$ ,  $PA = 80$  ft.,  $CA = 100$  ft. Let  $BP = x$  ft., then

$$\tan \theta = \frac{x+80}{100}, \quad \tan (\theta - \beta) = \frac{80}{100} = \frac{4}{5},$$

$$\therefore \frac{\tan \theta - \tan \beta}{1 + \tan \theta \tan \beta} = \frac{4}{5},$$

$$\frac{\frac{x+80}{100} - \frac{1}{9}}{1 + \frac{x+80}{900}} = \frac{4}{5};$$

$$\therefore 5(9x + 720 - 100) = 4(980 + x);$$

$$45x + 3100 = 3920 + 4x;$$

$$41x = 820; \text{ or } x = 20.$$

$$215. (1) \cot^{-1} 7 + \cot^{-1} 8 = \cot^{-1} \frac{7 \cdot 8 - 1}{7 + 8} = \cot^{-1} \frac{55}{15},$$

$$\cot^{-1} 3 - \cot^{-1} 18 = \cot^{-1} \frac{3 \cdot 18 + 1}{18 - 3} = \cot^{-1} \frac{55}{15}.$$

$$(2) 4 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{5^2}} = 2 \tan^{-1} \frac{5}{12}$$

$$= \tan^{-1} \frac{\frac{2 \times 5}{12}}{1 - \frac{5^2}{12^2}} = \tan^{-1} \frac{120}{119}.$$

$$\therefore 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \tan^{-1} \frac{120 \cdot 239 - 119}{119 \cdot 239 + 120}$$

$$= \tan^{-1} \frac{119 \cdot 239 + (239 - 119)}{119 \cdot 239 + 120} = \tan^{-1} 1 = \frac{\pi}{4}.$$

216. Proceeding as in Art. 259, Ex. 2, we find that  $\frac{A}{2}$  lies between  $2n\pi + \frac{3\pi}{4}$  and  $2n\pi + \frac{5\pi}{4}$ ; that is  $A$  lies between  $(8n+3)\frac{\pi}{2}$  and  $(8n+5)\frac{\pi}{2}$ .

$$217. \text{ First side } = \frac{1}{2} \left\{ 1 - \cos \left( \frac{\pi}{4} + \theta \right) - 1 + \cos \left( \frac{\pi}{4} - \theta \right) \right\}$$

$$= \sin \frac{\pi}{4} \sin \theta = \frac{1}{\sqrt{2}} \sin \theta.$$

218. From the two given relations we easily deduce

$$x = \frac{\sin \theta}{\sin (\theta + \phi)}, \quad y = \frac{\sin \phi}{\sin (\theta + \phi)},$$

$$\therefore \sin \theta : \sin \phi = x : y.$$

$$219. \tan^{-1} \frac{x+1+x-1}{1-(x^2-1)} = \tan^{-1} \frac{8}{31};$$

$$\therefore \frac{2x}{2-x^2} = \frac{8}{31}, \text{ or } 4x^2 + 31x - 8 = 0;$$

$$\therefore (4x-1)(x+8) = 0, \text{ or } x = \frac{1}{4}, \text{ or } -8.$$

Again  $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$   
 $= 1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3)$   
 $= 1 + 4 + 1 + 9 = 15.$

220. Let  $5A = \alpha$ ,  $5B = \beta$ ,  $5C = \gamma$ , then  $\alpha + \beta + \gamma = 5\pi$ ;

$$\therefore \sin(\alpha + \beta) = \sin(\pi - \gamma) = \sin \gamma; \cos \gamma = -\cos(\alpha + \beta),$$

$$\begin{aligned} \sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 2 \sin(\alpha + \beta) \cos(\alpha - \beta) + 2 \sin \gamma \cos \gamma \\ &= 2 \sin \gamma \{\cos(\alpha - \beta) - \cos(\alpha + \beta)\} \\ &= 4 \sin \alpha \sin \beta \sin \gamma. \end{aligned}$$

In the second case the sum of the three angles is  $\frac{16\pi}{2^3}$ , or  $\frac{\pi}{2}$  and the result easily follows as in Art. 135, Ex. 2.

221. See solution to XVIII. a. Ex. 24.

222. We have  $x = \frac{BD \sin 15^\circ}{\sin 50^\circ}$ ;  $BD = \frac{100}{\cos 25^\circ}$ ;

$$\therefore x = \frac{100 \cos 75^\circ}{\cos 40^\circ \cos 25^\circ};$$

$$\begin{aligned} \therefore \log x &= 2 + \log \cos 75^\circ - (\log \cos 40^\circ + \log \cos 25^\circ) \\ &= 1.4129962 - (1.8415297) \\ &= 1.5714665 \end{aligned}$$

$$\begin{array}{r} \log 37.279 = 1.5714643 \\ \text{diff.} \quad \quad \quad 22 \end{array}$$

$$\begin{aligned} \text{prop}^l. \text{ increase} &= \frac{22}{116} \times .001 \\ &= .00019; \end{aligned}$$

$$\therefore x = 37.27919.$$

$$\begin{aligned} 223. \quad \{\sec \theta + \operatorname{cosec} \theta (1 + \sec \theta)\}^2 &= \left( \frac{1}{\cos \theta} + \frac{1 + \cos \theta}{\sin \theta \cos \theta} \right)^2 \\ &= \frac{(1 + \sin \theta + \cos \theta)^2}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{2 + 2 \sin \theta + 2 \cos \theta + 2 \sin \theta \cos \theta}{\sin^2 \theta \cos^2 \theta}; \end{aligned}$$

$$\begin{aligned} \therefore \text{First side} &= 2 \sec^2 \theta \frac{(1 - \cos \theta)}{\sin^2 \theta} (1 + \sin \theta + \cos \theta + \sin \theta \cos \theta) \\ &= \frac{2 \sec^2 \theta (1 + \cos \theta) (1 + \sin \theta)}{1 + \cos \theta} \\ &= 2 \sec^2 \theta (1 + \sin \theta). \end{aligned}$$

224. The relation given will be true if

$$\frac{1}{a+b+c} - \frac{1}{a+c} = \frac{1}{b+c} - \frac{2}{a+b+c};$$

$$\text{i.e. if } -\frac{b}{a+c} = \frac{a-b-c}{b+c}, \text{ or } \frac{b}{a+c} = \frac{b+c-a}{b+c},$$

i.e. if

$$b(b+c) = (b+c)(a+c) - a^2 - ac.$$

From this we easily deduce  $\frac{a^2+b^2-c^2}{ab} = 1$ , which is true when  $C = 60^\circ$ .

225. The solution of this example is merely an extension of that of Ex. 205.

226. We have  $m \sin(\alpha - \theta) \cos(\alpha - \theta) = n \sin \theta \cos \theta$ ,

$$m \sin 2(\alpha - \theta) = n \sin 2\theta;$$

$$\therefore \frac{\sin 2(\alpha - \theta) - \sin 2\theta}{\sin 2(\alpha - \theta) + \sin 2\theta} = \frac{n - m}{n + m},$$

$$\frac{\sin(\alpha - 2\theta) \cos \alpha}{\cos(\alpha - 2\theta) \sin \alpha} = \frac{n - m}{n + m},$$

$$\tan(\alpha - 2\theta) = \frac{n - m}{n + m} \tan \alpha,$$

$$\alpha - 2\theta = \tan^{-1} \left( \frac{n - m}{n + m} \tan \alpha \right),$$

$$\theta = \frac{1}{2} \left\{ \alpha - \tan^{-1} \left( \frac{n - m}{n + m} \tan \alpha \right) \right\}.$$

227. Put  $k$  for each of the equal ratios, then it easily follows that  $s = k(1 + n^2)$ .

Now  $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(1-m^2)m^2(1+n^2)}{(1+n^2)n^2(1-m^2)}} = \frac{m}{n};$

$$\therefore A = 2 \tan^{-1} \frac{m}{n}; \text{ similarly } B = 2 \tan^{-1} mn.$$

Again,

$$\begin{aligned} \Delta &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= k^2 \sqrt{(1+n^2)^2 (1-m^2)^2 m^2 n^2} \\ &= k^2 (1-m^2)(1+n^2)mn = kmn \\ &= \frac{mnb}{m^2+n^2}, \text{ since } \frac{b}{m^2+n^2} = k. \end{aligned}$$

228. See figure and solution of Example II. page 190.

$$\begin{aligned}\text{Here} \quad h &= CD = \frac{a \sin \beta}{\cos (2\alpha + \beta)}, \\ l &= DE = \frac{a \sin \alpha \cos (\alpha + \beta)}{\cos (2\alpha + \beta)}.\end{aligned}$$

$$\begin{aligned}\text{But} \quad 2\alpha + \beta + \theta &= 90^\circ; \therefore \cos (2\alpha + \beta) = \sin \theta; \\ \therefore \text{by substitution,} \quad h &= a \sin \beta \operatorname{cosec} \theta, \\ 2l &= 2a \operatorname{cosec} \theta \sin \alpha \cos (\alpha + \beta) \\ &= a \operatorname{cosec} \theta \{ \sin (2\alpha + \beta) - \sin \beta \} \\ &= a \operatorname{cosec} \theta (\cos \theta - \sin \beta).\end{aligned}$$

229. We have

$$\begin{aligned}\frac{1}{2} \tan \frac{\theta}{2} + \cot \theta \\ &= \frac{1}{2} \cdot \frac{1 - \cos \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \frac{1 - \cos \theta + 2 \cos \theta}{2 \sin \theta} = \frac{1 + \cos \theta}{2 \sin \theta} \\ &= \frac{2 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\ &= \frac{\cos^2 \frac{\theta}{4} - \sin^2 \frac{\theta}{4}}{4 \sin \frac{\theta}{4} \cos \frac{\theta}{4}} = \frac{1}{4} \cot \frac{\theta}{4} - \frac{1}{4} \tan \frac{\theta}{4}.\end{aligned}$$

230. With the figure of Art. 214 we have  $AD = \frac{AO}{2m}$ .

Let  $\theta = \angle AOD$ , then  $\cos AOB = 1 - 2 \sin^2 \theta$

$$= 1 - \frac{2AD^2}{AO^2} = 1 - \frac{1}{2m^2} = \frac{2m^2 - 1}{2m^2};$$

$$\therefore AOB = \sec^{-1} \frac{2m^2}{2m^2 - 1}.$$

231. With the figure of Art 268, Ex. 1, we have

$$\frac{PN}{ON} = \tan 1'' = \text{radian measure of } 1'', \text{ approx.}$$

$$= \frac{\pi}{180} \times \frac{1}{60} \times \frac{1}{60};$$

$$\begin{aligned}\therefore ON &= \frac{180 \times 60 \times 60}{\pi} \text{ inches} \\ &= \frac{180 \times 60 \times 60}{1760 \times 3 \times 12\pi} \text{ miles} \\ &= \frac{1800}{176\pi} \text{ miles} = 3\frac{1}{4} \text{ miles, nearly.}\end{aligned}$$

232.

$$\tan^{-1} y = 2 \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{\frac{4x}{1-x^2}}{1 - \frac{4x^2}{(1-x^2)^2}}$$

$$= \tan^{-1} \frac{4x(1-x^2)}{1-6x^2+x^4};$$

$$\therefore y = \frac{4x(1-x^2)}{1-6x^2+x^4}.$$

If

$$y = \tan \frac{\pi}{2}, \quad 1-6x^2+x^4=0,$$

but

$$x = \tan \frac{1}{4} (\tan^{-1} y) = \tan \frac{\pi}{8},$$

thus  $\tan \frac{\pi}{8}$  is a root of  $x^4 - 6x^2 + 1 = 0$ .

233. We have

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a} = \frac{49}{529};$$

$$\therefore \tan a = \pm \frac{7}{23}.$$

The two values may be explained as in Art. 261, Ex. 2.

234. We have

$$\frac{\sin \theta}{a} = \frac{\sin \phi}{b} = \frac{\sin (\theta + \phi)}{c};$$

$$\therefore \frac{\sin \theta + \sin \phi}{a + b} = \frac{\sin (\theta + \phi)}{c}.$$

But

$$a + b = 2c,$$

$$\therefore \sin \theta + \sin \phi = 2 \sin (\theta + \phi),$$

whence

$$\cos \frac{\theta - \phi}{2} = 2 \cos \frac{\theta + \phi}{2} \dots \dots \dots (1).$$

or

$$\begin{aligned}\cos \frac{\theta + \phi}{2} &= \cos \frac{\theta - \phi}{2} - \cos \frac{\theta + \phi}{2} \\ &= 2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \dots \dots \dots (2).\end{aligned}$$

Now  $\cos \theta + \cos \phi = 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2}$

$$= 2 \left( 2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \right) 2 \cos \frac{\theta + \phi}{2}, \text{ by (1) and (2)}$$

$$= 4 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \cdot 4 \sin \frac{\theta}{2} \sin \frac{\phi}{2}, \text{ by (2)}$$

$$= 16 \sin^2 \frac{\theta}{2} \sin^2 \frac{\phi}{2}$$

$$= 4 (1 - \cos \theta) (1 - \cos \phi).$$

235. (1)  $\sin 7\theta + \sin \theta = \sin 4\theta;$   
 $\therefore 2 \sin 4\theta \cos 3\theta = \sin 4\theta;$   
 $\therefore \text{either } \sin 4\theta = 0, \text{ or } \cos 3\theta = \frac{1}{2};$

that is,  $4\theta = n\pi, \text{ or } 3\theta = 2n\pi \pm \frac{\pi}{3}.$

(2)  $\tan x - \frac{\sqrt{3}}{\tan x} + 1 - \sqrt{3} = 0;$   
 $\tan^2 x - (\sqrt{3} - 1) \tan x - \sqrt{3} = 0;$   
 $(\tan x - \sqrt{3})(\tan x + 1) = 0;$   
 $\therefore \text{either } \tan x = \sqrt{3}, \text{ or } \tan x = -1;$

that is,  $x = n\pi + \frac{\pi}{3}, \text{ or } x = n\pi + \frac{3\pi}{4}.$

236. (1)  $\sin 3A = \sin 3(180^\circ - \overline{B + C})$   
 $= \sin (360^\circ + 180^\circ - 3\overline{B + C}) = \sin 3\overline{B + C}.$

We have only now to prove that

$$\Sigma \sin 3(B + C) \sin (B - C) = 0,$$

and this follows by separating each term into the difference of two cosines.

(2) It will be sufficient to prove that

$$\Sigma \sin^3 A \sin (B - C) = 0.$$

Now  $\sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A;$

$$\therefore \Sigma \sin^3 A \sin (B - C) = \frac{3}{4} \Sigma \sin A \sin \overline{B - C} - \frac{1}{4} \Sigma \sin 3A \sin \overline{B - C}$$

$$= 0,$$

by the first part of the question.

237. The angle  $ACP = \theta - A$ .

$$\therefore \frac{AP}{PC} = \frac{\sin(\theta - A)}{\sin A} = \sin \theta \cot A - \cos \theta,$$

$$\frac{PC}{PB} = \frac{\sin B}{\sin(\theta + B)} = \frac{1}{\sin \theta \cot B + \cos \theta};$$

$\therefore$  by multiplication

$$\frac{AP}{PB} = \frac{m}{n} = \frac{\sin \theta \cot A - \cos \theta}{\sin \theta \cot B + \cos \theta};$$

whence

$$\sin \theta (n \cot A - m \cot B) = (m + n) \cos \theta,$$

or

$$(m + n) \cot \theta = n \cot A - m \cot B.$$

238. The equation may be written

$$a \sin \theta - \cos \theta + b = 0 \dots\dots\dots (1).$$

Since  $\alpha$  and  $\beta$  are roots of this equation

$$a \sin \alpha - \cos \alpha + b = 0,$$

$$a \sin \beta - \cos \beta + b = 0,$$

whence  $a$  and  $b$  may be found.

Again from (1),

$$(a \sin \theta + b)^2 = 1 - \sin^2 \theta,$$

or

$$(1 + a^2) \sin^2 \theta + 2ab \sin \theta + b^2 - 1 = 0;$$

since  $\alpha, \beta$  are roots of this equation,

$$\sin \alpha + \sin \beta = -\frac{2ab}{1 + a^2}.$$

Similarly we may shew that  $\cos \alpha + \cos \beta = \frac{2b}{1 + a^2}$ , whence the required result follows.

239. Write  $s$  and  $c$  for  $\sin \theta$  and  $\cos \theta$  respectively; then

$$\frac{u_3 - u_5}{u_1} = \frac{s^3 + c^3 - (s^5 + c^5)}{s + c} = \frac{s^3(1 - s^2) + c^3(1 - c^2)}{s + c} = \frac{s^3c^2 + c^3s^2}{s + c} = s^2c^2.$$

$$\text{Again } \frac{u_5 - u_7}{u_3} = \frac{s^5 + c^5 - (s^7 + c^7)}{s^3 + c^3} = \frac{s^5(1 - s^2) + c^5(1 - c^2)}{s^3 + c^3}$$

$$= \frac{s^5c^2 + c^5s^2}{s^3 + c^3} = s^2c^2.$$

240. Let  $E, F$  be the first and second points of observation respectively; then  $EF = a$ , and  $EAD$  is a straight line. Let  $x =$  a side of the square base, then  $EA = AB = AD = x$ . Then if  $\angle AFD = \theta$ , we have

$$AD^2 = AF^2 + FD^2 - 2AF \cdot FD \cos \theta \dots\dots\dots (1).$$

But  $AD^2 = x^2$ ,  $AF^2 = x^2 + a^2$ ,  $FD^2 = 4x^2 + a^2$ . Also  $\cos \theta = \frac{1}{3}\sqrt{8}$ . Substituting these values in (1) we obtain  $x = \frac{a\sqrt{2}}{2}$ .

$$241. \quad (1) \quad \text{First side} = \frac{1 - \sin^2 A}{\sin A} \cdot \frac{1 - \cos^2 A}{\cos A} = \frac{\cos^2 A \sin^2 A}{\sin A \cos A} \\ = \sin A \cos A.$$

$$\text{Again, second side} = \left( \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \right)^{-1} = \sin A \cos A.$$

$$(2) \quad \text{First side} = \frac{\tan \theta}{\sec^4 \theta} + \frac{\cot \theta}{\operatorname{cosec}^4 \theta} \\ = \sin \theta \cos^3 \theta + \cos \theta \sin^3 \theta \\ = \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) \\ = \frac{1}{2} \sin 2\theta.$$

$$242. \quad \text{We have} \quad 2 \sin 4\theta \cos \theta = \frac{1}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2};$$

$$\therefore \sin 5\theta + \sin 3\theta = \frac{1}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2};$$

$$\therefore \sin 3\theta = \frac{1}{2},$$

one solution of which is  $3\theta = 30^\circ$ .

$$243. \quad \tan^{-1} \frac{2mn}{m^2 - n^2} = \tan^{-1} \frac{2 \frac{n}{m}}{1 - \frac{n^2}{m^2}} = 2 \tan^{-1} \frac{n}{m};$$

$$\therefore \text{first side} = 2 \tan^{-1} \frac{n}{m} + 2 \tan^{-1} \frac{q}{p} \\ = 2 \left( \tan^{-1} \frac{n}{m} + \tan^{-1} \frac{q}{p} \right) \\ = 2 \tan^{-1} \frac{\frac{n}{m} + \frac{q}{p}}{1 - \frac{nq}{mp}};$$

∴ first side

$$= 2 \tan^{-1} \frac{np + mq}{mp - nq} = 2 \tan^{-1} \frac{N}{M}$$

$$= \tan^{-1} \frac{2MN}{M^2 - N^2}.$$

244. Write  $\frac{r}{s-a}$  for  $\tan \frac{A}{2}$ , then the first side becomes

$$\frac{r}{(s-a)(a-b)(a-c)} + \text{two similar terms.}$$

Now

$$\frac{r}{(s-a)(a-b)(a-c)} = -\frac{rs(b-c)(s-b)(s-c)}{\Delta^2(a-b)(b-c)(c-a)}$$

$$= -\frac{1}{\Delta} \frac{(b-c)\{s^2 - s(b+c) + bc\}}{(a-b)(b-c)(c-a)}.$$

Now  $\Sigma(b-c)\{s^2 - s(b+c) + bc\} = -(a-b)(b-c)(c-a)$ .  
 [See Hall and Knight's Elem. Algebra, Art. 224.]

Thus the first side reduces to  $\frac{1}{\Delta}$ .

245. We have  $\left(1 - \frac{s-b}{s-a}\right) \left(1 - \frac{s-c}{s-a}\right) = 2;$

or

$$(b-a)(c-a) = 2(s-a)^2;$$

$$bc - ac - ab + a^2 = 2s^2 - 4as + 2a^2;$$

$$\therefore bc = 2s^2 - 4as + a(a+b+c)$$

$$= 2s^2 - 4as + 2as;$$

whence

$$\frac{s(s-a)}{bc} = \frac{1}{2};$$

that is,

$$\cos \frac{A}{2} = \frac{1}{\sqrt{2}}, \text{ or } A = 90^\circ.$$

246. Let  $A = 58^\circ 40' 3.9''$ ,  $b = 237$ ,  $c = 158$ .

Then as in Art. 197 we obtain

$$a = (b+c) \sin \theta, \text{ where } \cos \theta = \frac{2\sqrt{bc}}{b+c} \cos \frac{A}{2},$$

$$\cos \theta = \frac{2\sqrt{237 \times 158}}{395} \cos 29^\circ 20' 1.95''$$

$$= \frac{2\sqrt{6}}{5} \cos 29^\circ 20' 1.95'',$$

$$\begin{array}{rcl}
 \log 2 & = & \cdot 3010300 \\
 \log 3 & = & \cdot 4771213 \\
 & 2 & \overline{) \cdot 7781513} \\
 \frac{1}{2} \log 6 & = & \cdot 3890757 \\
 2 \log 2 - 1 & = & \bar{1} \cdot 6020600 \\
 \log \cos 29^\circ 20' & = & \bar{1} \cdot 9404091 \\
 \text{Subtract diff. for } 1 \cdot 95'' & & 23 \\
 \log \cos \theta & = & \bar{1} \cdot 9315425 \\
 \log \cos 31^\circ 20' & = & \bar{1} \cdot 9315374 \\
 & & \underline{51} \\
 \therefore \theta & = & 31^\circ 19' 56''.
 \end{array}$$

$$\frac{51}{769} \times 60'' = 3 \cdot 97''.$$

Again

$$\begin{array}{rcl}
 \log 395 & = & 2 \cdot 5965971 \\
 \log \sin 31^\circ 19' & = & \bar{1} \cdot 7158092 \\
 \text{diff. for } 56'' & & 1937 \\
 \log a & = & 2 \cdot 3126000 \\
 \log 205 \cdot 4 & = & 2 \cdot 3126004 \\
 \therefore a & = & 205 \cdot 4.
 \end{array}$$

247. We have

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{3 \sin A}{\cos A};$$

whence

$$\sin(A+B) \cos A = 3 \cos(A+B) \sin A,$$

or

$$\sin(2A+B) + \sin B = 3 \sin(2A+B) - 3 \sin B;$$

that is

$$2 \sin(2A+B) = 4 \sin B.$$

Multiply by  $\cos B$ ; then by separating the product on the left into the sum of two sines we obtain the required result.

$$\begin{aligned}
 248. \quad \text{First side} &= 2 \sin(\theta - \alpha) \{ \sin(\theta - \alpha) + \sin(2m\theta - \alpha - \theta) \} \\
 &= 2 \sin^2(\theta - \alpha) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2\alpha) \\
 &= 1 - \cos(2\theta - 2\alpha) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2\alpha).
 \end{aligned}$$

249. Let  $AD$  be perpendicular to  $BC$  and meet the circum-circle in  $E$ ; then  $\angle BED = C$ , and  $\alpha = DE$ .

Now

$$\frac{BD}{\alpha} = \tan C, \text{ and } \frac{DC}{\alpha} = \tan B;$$

$$\therefore \frac{BD+DC}{\alpha} = \frac{\alpha}{\alpha} = \tan B + \tan C.$$

$$\text{Similarly } \frac{b}{\beta} = \tan C + \tan A, \quad \frac{c}{\gamma} = \tan A + \tan B;$$

whence the result follows.

250. From the first equation,  $3 \sin^2 A = 1 - 2 \sin^2 B - \cos 2B$ ; and from the second equation,

$$6 \sin A \cos A - 2 \sin 2B = 0;$$

multiply each term by  $\sin A$ , and we have

$$3 \sin^2 A \cos A - \sin 2B \sin A = 0.$$

Substituting  $\cos 2B$  for  $3 \sin^2 A$ , we obtain

$$\cos 2B \cos A - \sin 2B \sin A = 0;$$

that is,

$$\cos (A + 2B) = 0; \text{ or } A + 2B = 90^\circ.$$

$$251. (1) \text{ First side} = \cot^{-1} \left( \frac{1}{\cot 2x} \right) + \cot^{-1} \left( -\frac{1}{\cot 3x} \right)$$

$$= \cot^{-1} \frac{\frac{1}{\cot 2x} \left( -\frac{1}{\cot 3x} \right) - 1}{-\frac{1}{\cot 3x} + \frac{1}{\cot 2x}}$$

$$= \cot^{-1} \left( \frac{\cot 3x \cot 2x + 1}{\cot 2x - \cot 3x} \right)$$

$$= \cot^{-1} (\cot x) = x.$$

$$(2) \text{ First side} = \tan^{-1} \frac{\frac{1-x}{1+x} - \frac{1-y}{1+y}}{1 + \frac{(1-x)(1-y)}{(1+x)(1+y)}} = \tan^{-1} \frac{2(y-x)}{2(1+xy)}$$

$$= \sin^{-1} \frac{\frac{y-x}{1+xy}}{\sqrt{1 + \frac{(y-x)^2}{(1+xy)^2}}} = \sin^{-1} \frac{y-x}{\sqrt{1+x^2} \sqrt{1+y^2}}.$$

252. See Art. 197.

Let  $A$  be the position of the station,  $B$  and  $C$  the positions of the two points; then  $A = 49^\circ 45'$ ,  $c = 1250$  yds.,  $b = 1575$  yds.

$$\text{Now } a = 2825 \cos \theta, \text{ where } \sin \theta = \frac{2 \sqrt{1250 \times 1575}}{2825} \cos 24^\circ 52' 30'',$$

$$\begin{array}{r} \log 1250 \\ \log 1575 \end{array} \quad \begin{array}{r} = 3.0969100 \\ = 3.1972806 \end{array}$$

$$2 \mid 6.2941906$$

$$3.1470953$$

$$\begin{array}{r} \log 2 \\ \log \cos 24^\circ 52' 30'' \end{array} \quad \begin{array}{r} = .3010300 \\ = 1.9577163 \end{array}$$

$$3.4058416$$

$$\log 2825 = 3.4510185$$

$$\log \sin \theta = 1.9548231 = \log \sin 64^\circ 19'.$$

$$\begin{array}{r} \log \cos 24^\circ 52' = 1.9577456 \\ \text{subtract } \frac{1}{2} \times 586 \quad \underline{293} \\ 1.9577163 \end{array}$$

Again

$$\begin{array}{rcl}
 \log 2825 & = & 3.4510185 \\
 \log \cos 64^\circ 19' & = & \bar{1}.6368859 \\
 \log a & = & 3.0879044 \\
 \log 1224.3 & = & 3.0878978 \\
 & & \hline
 & & 166 \\
 & 4 & 142 \\
 & & \hline
 & & 240 \\
 & 7 & 249 \\
 & & \hline
 \end{array}$$

$$\therefore a = 1224.347 \text{ yards.}$$

254. Multiply all through by 2; then

First side  $= 1 + \cos 2S + 1 + \cos 2(S - A) + \text{two similar terms}$ 

$$= 4 + 2 \cos (2S - A) \cos A + 2 \cos (2S - B - C) \cos (B - C)$$

$$= 4 + 2 \cos (B + C) \cos A + 2 \cos A \cos (B - C)$$

$$= 4 + 4 \cos A \cos B \cos C.$$

255. It is easy to see that this is the same as Example 1 in Art. 135.

256. We have  $R = \frac{a}{2 \sin A} = 18 \operatorname{cosec} 61^\circ 15'.$ 

$$\begin{array}{rcl}
 \log 18 & = & 1.2552725 \\
 \log \operatorname{cosec} 61^\circ 15' & = & .0571357 \\
 \log R & = & 1.3124082 \\
 \log 20.530 & = & 1.3123889 \\
 & & \hline
 & & 193 \\
 & 9 & 191 \\
 & & \hline
 \end{array}$$

$$\therefore R = 20.5309.$$

Again,

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$\begin{array}{rcl}
 \log R & = & 1.3124082 \\
 \log 4 & = & .6020600 \\
 \log \sin 30^\circ 37' & = & \bar{1}.7069667 \\
 \text{diff. for } 30'' & = & 1067 \\
 \log \sin 30^\circ 37' & = & \bar{1}.7755801 \\
 \text{diff. for } 30'' & = & 850 \\
 \log \sin 22^\circ 45' & = & \bar{1}.5873865 \\
 \therefore \log r & = & .9845932 \\
 \log 9.6514 & = & .9845903 \\
 & & \hline
 & & 29 \\
 & 6 & 27 \\
 & & \hline
 \end{array}$$

$$\therefore r = 9.65146.$$

257. This follows from XVIII. c. Ex. 5 and XII. d. Ex. 12.

258. Let

$$\angle APB = \alpha, \angle BPC = \beta, \angle PBC = \gamma;$$

$$\frac{PB}{AB} = \frac{\sin(\gamma - \alpha)}{\sin \alpha}, \quad \frac{PB}{BC} = \frac{\sin(\beta + \gamma)}{\sin \beta};$$

then

but

$$AB = BC,$$

$$\therefore \frac{\sin(\gamma - \alpha)}{\sin \alpha} = \frac{\sin(\beta + \gamma)}{\sin \beta},$$

or

$$\sin \gamma \cot \alpha - \cos \gamma = \cos \gamma + \sin \gamma \cot \beta;$$

$$\therefore 2 \cos \gamma = \sin \gamma (\cot \alpha - \cot \beta),$$

$$2 \cot \gamma = \cot \alpha - \cot \beta;$$

that is,

$$\frac{2}{T} = \frac{1}{t'} - \frac{1}{t},$$

 since  $\gamma$  is the supplement of the angle  $BP$  makes with the road.

$$259. \text{ First side} = \frac{(\cos B + \cos C)(1 + 2 \cos A)}{(1 + 2 \cos A)(1 - \cos A)} = \frac{2 \cos \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \sin^2 \frac{A}{2}}$$

$$= \frac{2 \cos \frac{B-C}{2}}{2 \sin \frac{A}{2}} \cdot \frac{\cos \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2 \cos \frac{B-C}{2} \sin \frac{B+C}{2}}{\sin A}$$

$$= \frac{\sin B + \sin C}{\sin A} = \frac{b+c}{a}.$$

$$260. \text{ From the fig. of Art. 219, we have } \frac{AI}{AI_1} = \frac{s-a}{s};$$

$$\therefore \text{ first side} = \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s-2s}{s} = 1.$$

$$261. \text{ Let } \sin^{-1} \frac{1}{3} = \alpha, \text{ then } \cos \alpha = \frac{2\sqrt{2}}{3};$$

$$\sin^{-1} \frac{3}{\sqrt{11}} = \beta, \text{ then } \cos \beta = \frac{\sqrt{2}}{\sqrt{11}}.$$

$$\text{Now } \sin(\alpha + \beta) = \frac{1}{3} \cdot \frac{\sqrt{2}}{\sqrt{11}} + \frac{2\sqrt{2}}{3} \cdot \frac{3}{\sqrt{11}} = \frac{7\sqrt{2}}{3\sqrt{11}}$$

$$= \cos \left( \sin^{-1} \frac{1}{3\sqrt{11}} \right),$$

$$\therefore \alpha + \beta = \frac{\pi}{2} - \sin^{-1} \frac{1}{3\sqrt{11}},$$

$$\sin^{-1} \frac{1}{3} + \sin^{-1} \frac{3}{\sqrt{11}} + \sin^{-1} \frac{3}{\sqrt{11}} = \frac{\pi}{2}.$$

or

262. We have  $\cot \alpha (\cot \beta \cot \gamma - 1) = \cot \beta + \cot \gamma$ ,

$$\begin{aligned}\therefore \cot (\beta + \gamma) &= \frac{1}{\cot \alpha} = \frac{1}{\tan \alpha} \\ &= \cot \left( \frac{\pi}{2} - \alpha \right); \end{aligned}$$

$$\therefore \beta + \gamma = n\pi + \frac{\pi}{2} - \alpha,$$

or

$$\alpha + \beta + \gamma = (2n + 1) \frac{\pi}{2}.$$

263. Since  $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ ,

$$\begin{aligned} \text{the first side} &= \frac{\tan^2 A + \tan^2 B + \tan^2 C}{\tan A + \tan B + \tan C} \\ &= \frac{(\tan A + \tan B + \tan C)^2 - 2 \tan A \tan B - 2 \tan B \tan C - 2 \tan C \tan A}{\tan A + \tan B + \tan C} \\ &= \tan A + \tan B + \tan C - \frac{2(\tan A \tan B + \dots + \dots)}{\tan A \tan B \tan C} \\ &= \tan A + \tan B + \tan C - 2(\cot A + \cot B + \cot C). \end{aligned}$$

264. Let  $O_1, O_2$  be the two points of observation,  $A$  and  $B$  the two objects, so that  $\angle A O_1 O_2 = 45^\circ$ ,  $\angle A O_2 O_1 = \angle O_1 O_2 B = 22\frac{1}{2}^\circ$ . Then  $\angle O_1 A O_2 = 112\frac{1}{2}^\circ$ ,  $\angle O_1 B O_2 = 22\frac{1}{2}^\circ$ , and  $O_1 B = O_1 O_2 = 1$  mile.

$$\text{Now from the } \triangle O_1 A O_2, \quad \frac{O_1 A}{1 \text{ mile}} = \frac{\sin 22\frac{1}{2}^\circ}{\sin 112\frac{1}{2}^\circ} = \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1;$$

$$\therefore O_1 A = \sqrt{2} - 1 \text{ miles}; \quad \therefore AB = \sqrt{2} \text{ miles.}$$

Again, if  $p_1, p_2$  be the perpendiculars from  $AB$  on  $O_1 O_2$

$$p_1 + p_2 = (O_1 A + O_1 B) \sin 45^\circ = AB \sin 45^\circ = 1 \text{ mile.}$$

265. This follows from the identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

where  $A + B + C = 180^\circ$ , by putting  $A = 20^\circ$ ,  $B = 40^\circ$ ,  $C = 120^\circ$ .

266. The equation may be written

$$(2 \operatorname{cosec} 2\theta)^3 = 3 (2 \operatorname{cosec} 2\theta) + \frac{\cos^3 \theta}{\sin^3 \theta},$$

or

$$\left( \frac{1}{\sin \theta \cos \theta} \right)^3 = \frac{3}{\sin \theta \cos \theta} + \frac{\cos^3 \theta}{\sin^3 \theta};$$

that is,

$$1 = 3 \sin^2 \theta \cos^2 \theta + \cos^6 \theta,$$

which reduces to  $(\cos^2 \theta - 1)^3 = 0$ , whence  $\theta = n\pi$ .

267. When  $A + B + C = 180^\circ$ ,

$$\begin{aligned}
 1 - \cos^2 B + \cos^2 A - \cos^2 C &= \sin^2 B + \sin^2 C - \sin^2 A \\
 &= \sin^2 B + \sin(C + A) \sin(C - A) \\
 &= \sin B \{ \sin(C + A) + \sin(C - A) \}, \\
 &\qquad\qquad\qquad \text{since } C + A = 180^\circ - B, \\
 &= 2 \sin B \sin C \cos A.
 \end{aligned}$$

When  $A + B + C = 0$ ,

$$\begin{aligned}
 1 - \cos^2 B + \cos^2 A - \cos^2 C &= \sin^2 B + \sin(C + A) \sin(C - A) \\
 &= -\sin B \{ \sin(C + A) + \sin(C - A) \}, \\
 &\qquad\qquad\qquad \text{since } C + A = -B, \\
 &= -2 \sin B \sin C \cos A.
 \end{aligned}$$

268. We have  $\cot A - \cot B = \cot B - \cot C$ ,

that is  $\frac{\sin(A - B)}{\sin A \sin B} = \frac{\sin(B - C)}{\sin B \sin C}$ ,

or  $\frac{\sin(A - B)}{\sin(B + C)} = \frac{\sin(B - C)}{\sin(A + B)}$ .

Whence  $\sin(A + B) \sin(A - B) = \sin(B + C) \sin(B - C)$ ,

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C;$$

that is,  $a^2 - b^2 = b^2 - c^2$ .

269. Second side  $= 2 \cos \frac{5a - 2\beta - \gamma}{4} \left\{ \cos \frac{4\beta + 3\gamma - 3a}{4} + \cos \frac{6\beta - 7\gamma + a}{4} \right\}$

$$\begin{aligned}
 &= \cos \frac{2a + 2\beta + 2\gamma}{4} + \cos \frac{6\beta + 4\gamma - 8a}{4} + \cos \frac{6a + 4\beta - 8\gamma}{4} \\
 &\qquad\qquad\qquad + \cos \frac{6\gamma + 4a - 8\beta}{4} \\
 &= \cos \frac{\pi}{2} + \cos \left( \frac{3\beta}{2} + \gamma - 2a \right) + \cos \left( \frac{3a}{2} + \beta - 2\gamma \right) \\
 &\qquad\qquad\qquad + \cos \left( \frac{3\gamma}{2} + a - 2\beta \right) \\
 &= \text{first side.}
 \end{aligned}$$

270. Denote the radii of the three escribed circles by  $x, y, z$  respectively then we have to shew that

$$\begin{aligned}
 (y - z)(z - x)(x - y) + (y - z)(z + x)(x + y) + (z - x)(x + y)(y + z) \\
 + (x - y)(y + z)(z + x) = 0.
 \end{aligned}$$

Taking the terms in pairs, the expression on the left reduces to

$$(y - z) \{ 2(zx + xy) \} + (y + z) \{ 2(zx - xy) \},$$

or

$$2x(y - z)(y + z) + 2x(y + z)(z - y),$$

which is identically equal to zero.

271. We have  $32 \sin \frac{A}{2} \sin \frac{5A}{2} = 16 (\cos 2A - \cos 3A)$ .

Now  $\cos 2A = 2 \cos^2 A - 1 = \frac{2 \times 9}{16} - 1 = \frac{1}{8},$

$$\cos 3A = 4 \cos^3 A - 3 \cos A = \frac{4 \times 27}{64} - \frac{3 \times 3}{4} = -\frac{9}{16};$$

$$\therefore 32 \sin \frac{A}{2} \sin \frac{5A}{2} = 16 \left( \frac{1}{8} + \frac{9}{16} \right) = 11.$$

272. Solving the quadratic, we have  $\tan \theta = -1 \pm \sqrt{2}$ .

Now  $\sqrt{2} - 1 = \tan \frac{\pi}{8}. \quad [\text{Art. 251.}]$

$$-(\sqrt{2} + 1) = -\cot \frac{\pi}{8} = -\tan \left( \frac{\pi}{2} - \frac{\pi}{8} \right).$$

From the first result, we get  $\theta = n\pi + \frac{\pi}{8},$

and from the second,  $\theta = n\pi - \left( \frac{\pi}{2} - \frac{\pi}{8} \right) = n\pi - \frac{3\pi}{8},$

both of which are included in  $(8n-1)\frac{\pi}{8} \pm \frac{\pi}{4}.$

$$273. (1) \quad 2 \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{2}{7}}{1 - \frac{1}{7^2}} = \tan^{-1} \frac{14}{48} = \tan^{-1} \frac{7}{24} = \cos^{-1} \frac{24}{25},$$

$$\begin{aligned} 4 \tan^{-1} \frac{1}{3} &= 2 \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} = 2 \tan^{-1} \frac{3}{4} = \tan^{-1} \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \tan^{-1} \frac{24}{7} \\ &= \sin^{-1} \frac{24}{25}. \end{aligned}$$

Thus each side of the identity  $= \frac{24}{25}.$

$$(2) \quad \frac{3 \sin 2\alpha}{5 + 3 \cos 2\alpha} = \frac{6 \tan \alpha}{1 + \tan^2 \alpha} \div \left\{ 5 + \frac{3(1 - \tan^2 \alpha)}{1 + \tan^2 \alpha} \right\} = \frac{3 \tan \alpha}{4 + \tan^2 \alpha};$$

$$\therefore \text{first side of the identity} = \tan^{-1} \frac{3 \tan \alpha}{4 + \tan^2 \alpha} + \tan^{-1} \left( \frac{\tan \alpha}{4} \right)$$

$$\begin{aligned} &= \tan^{-1} \frac{\frac{3 \tan \alpha}{4 + \tan^2 \alpha} + \frac{\tan \alpha}{4}}{1 - \frac{3 \tan^2 \alpha}{4(4 + \tan^2 \alpha)}} = \tan^{-1} \frac{16 \tan \alpha + \tan^3 \alpha}{16 + \tan^2 \alpha} \\ &= \tan^{-1} (\tan \alpha) = \alpha. \end{aligned}$$

274. We have

$$\frac{s(s-a)}{bc} = \frac{b^2 + c^2}{4bc};$$

$$\therefore 2s(2s-2a) = b^2 + c^2;$$

$$(b+c+a)(b+c-a) = b^2 + c^2,$$

$$b^2 + c^2 + 2bc - a^2 = b^2 + c^2;$$

$$\therefore \frac{a^2}{2} = bc,$$

or  
that is,

which proves the proposition.

275. We have  $\sin B = \frac{b}{a} \sin A = \frac{119 \sin 50^\circ}{97},$

$$\log 119 = 2.0755470$$

$$\log \sin 50^\circ = 1.8842540$$

$$1.9598010$$

$$\log 97 = 1.9867717$$

$$\log \sin B = 1.9730293$$

$$\log \sin 70^\circ = 1.9729858$$

$$435$$

$$\text{diff. for } 1' = 460,$$

$$\frac{435}{460} \times 60'' = 57'';$$

$\therefore B = 70^\circ 0' 57''$ , or  $109^\circ 59' 3''$ , both values being admissible since  $a < b$ ;  
 $\therefore C = 59^\circ 59' 3''$  or  $20^\circ 0' 57''$ .

276. The  $\angle BD_1F_1 = \frac{1}{2}$  (suppt of  $\angle F_1BD_1$ )  $= \frac{B}{2} = \angle BF_1D_1$ .

Similarly  $\angle CD_1E_1 = \frac{C}{2};$

$$\therefore \angle F_1D_1E_1 = 180^\circ - \frac{B+C}{2} = 90^\circ + \frac{A}{2}.$$

Again from the isosceles triangle  $AE_1F_1$ ,

$$\angle AF_1E_1 = 90^\circ - \frac{A}{2}; \therefore \angle D_1F_1E_1 = 90^\circ - \frac{A+B}{2} = \frac{C}{2}.$$

and similarly

$$\angle D_1E_1F_1 = \frac{B}{2}.$$

Now  $r_a = \frac{E_1F_1 \sin \frac{1}{2} E_1 \sin \frac{1}{2} F_1}{\cos \frac{1}{2} D_1} = \frac{2s \sin \frac{A}{2} \sin \frac{B}{4} \sin \frac{C}{4}}{\cos \left( 45^\circ + \frac{A}{4} \right)}$

$$= 4s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \cdot \frac{\cos \frac{A}{4}}{\frac{1}{\sqrt{2}} \left( \cos \frac{A}{4} - \sin \frac{A}{4} \right)}$$

$$= 4\sqrt{2}s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \cdot \frac{1}{1 - \tan \frac{A}{4}};$$

$$\begin{aligned}\therefore \frac{1}{r_a} : 1 - \tan \frac{A}{4} &= 1 : 4\sqrt{2}s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \\ &= \frac{1}{r_b} : 1 - \tan \frac{B}{4} = \frac{1}{r_c} : 1 - \tan \frac{C}{4}\end{aligned}$$

by symmetry.

277. The expression  $= \tan^{-1} \frac{x \cos \theta}{1 - x \sin \theta} - \tan^{-1} \frac{x - \sin \theta}{\cos \theta}$

and this reduces to  $\tan^{-1} \left\{ \frac{\sin \theta (1 - 2x \sin \theta + x^2)}{\cos \theta (1 - 2x \sin \theta + x^2)} \right\},$

which equals  $\tan^{-1}(\tan \theta)$ , or  $\theta$ .

278. By Example XIII. c. 8 we have

$$\frac{\cos A + \cos B}{4 \sin^2 \frac{C}{2}} = \frac{a + b}{2c};$$

$\therefore a + b = 2c$ ; whence  $a, c, b$  are in A.P.

279. The expression

$$\begin{aligned}&= 2 \cos \alpha \cos \beta \{ \cos (\gamma + \delta) + \cos (\gamma - \delta) \} + 2 \sin \alpha \sin \beta \{ \cos (\gamma - \delta) - \cos (\gamma + \delta) \} \\ &= 2 \cos (\gamma + \delta) \cos (\alpha + \beta) + 2 \cos (\gamma - \delta) \cos (\alpha - \beta) \\ &= \cos (\alpha + \beta + \gamma + \delta) + \cos (\alpha + \beta - \gamma - \delta) + \cos (\alpha - \beta + \gamma - \delta) + \cos (\alpha - \beta - \gamma + \delta).\end{aligned}$$

280. The  $\angle BIC = 180^\circ - \frac{B + C}{2} = 90^\circ + \frac{A}{2};$

$$\therefore \rho_1 = \frac{a}{2 \sin BIC} = \frac{a}{2 \cos \frac{A}{2}};$$

$$\begin{aligned}\therefore \rho_1 \rho_2 \rho_3 &= \frac{abc}{8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{R^3 \sin A \sin B \sin C}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \\ &= 8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 2rR^2,\end{aligned}$$

since

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

281. This is a particular case of Ex. 13 in XVII. a.

282. The equation may be written

$$b^2 \sin^2 2\theta = (c - a \cos 2\theta)^2,$$

or

$$b^2 (1 - \cos^2 2\theta) = c^2 - 2ac \cos 2\theta + a^2 \cos^2 2\theta,$$

that is,

$$(a^2 + b^2) \cos^2 2\theta - 2ac \cos 2\theta + c^2 - b^2 = 0;$$

therefore, by the theory of quadratic equations,

$$\cos 2\alpha + \cos 2\beta = \frac{2ac}{a^2 + b^2};$$

$$\therefore 2 \cos^2 \alpha - 1 + 2 \cos^2 \beta - 1 = \frac{2ac}{a^2 + b^2};$$

whence

$$\cos^2 \alpha + \cos^2 \beta = \frac{a^2 + ac + b^2}{a^2 + b^2}.$$

283. We have  $c^2 = a^2 + b^2 - 2ab \cos C$

$$= 2 + 2 + \sqrt{2} - 2\sqrt{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$= 4 + \sqrt{2} - \sqrt{2} (2 + \sqrt{2}) = 2 - \sqrt{2};$$

$$\therefore c = \sqrt{2 - \sqrt{2}}.$$

Now  $\sin A = \frac{a}{c} \sin C = \frac{\sqrt{2}}{\sqrt{2 - \sqrt{2}}}, \quad \frac{\sqrt{2 - \sqrt{2}}}{2} = \frac{1}{\sqrt{2}};$

therefore  $A = 45^\circ$ , or  $135^\circ$ , and since  $a$  is not the greatest side the smaller value must be taken.

Therefore

$$B = 112\frac{1}{2}^\circ.$$

284. We have  $\sin 3A = 3 \sin A - 4 \sin^3 A;$

$$\therefore \sin 3A = \frac{3}{4} \sin A - \frac{1}{4} \sin^3 A;$$

$$\therefore \Sigma \sin^3 A = \frac{3}{4} \Sigma \sin A - \frac{1}{4} \Sigma \sin 3A.$$

Now  $\Sigma \sin A = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},$

and  $\Sigma \sin 3A = 2 \sin \frac{3(A+B)}{2} \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2}$

$$= 2 \sin \left( 270^\circ - \frac{3C}{2} \right) \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2}$$

$$= -2 \cos \frac{3C}{2} \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2}$$

$$\begin{aligned}
&= 2 \cos \frac{3C}{3} \left\{ \sin \frac{3C}{2} - \cos \frac{3(A-B)}{2} \right\} \\
&= 2 \cos \frac{3C}{2} \left\{ -\cos \frac{3(A+B)}{2} - \cos \frac{3(A-B)}{2} \right\} \\
&= -4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2};
\end{aligned}$$

$$\therefore \Sigma \sin^3 A = 3 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}.$$

285. Take the third figure on p. 131, and first suppose that

$$\angle AB_2C = 2 \angle AB_1C.$$

Then it easily follows that  $\triangle CB_1B_2$  is equilateral;

$$\therefore b \sin A = a \sin B \text{ becomes } b \sin A = \frac{\sqrt{3}}{2} a.$$

Secondly, suppose that  $\angle ACB_1 = 2 \angle ACB_2$ .

Then  $\angle ACB_2 = \angle B_1 - \angle A = \angle B_2CB_1 = 180^\circ - 2 \angle B_1$ ;

$$\therefore 3B_1 = 180^\circ + A,$$

$$\sin 3B_1 + \sin A = 0; \text{ or } 3 \sin B_1 - 4 \sin^3 B_1 + \sin A = 0.$$

Substituting  $\frac{b}{a} \sin A$  for  $\sin B_1$ , and reducing we obtain the required result.

286. If we write  $\sqrt{y}$  in the place of  $x$  the resulting equation has  $\tan^2 \alpha$ ,  $\tan^2 \beta$ ,  $\tan^2 \gamma$  for its roots. If in the last equation we further write  $z$  for  $1+y$  the resulting equation has  $\sec^2 \alpha$ ,  $\sec^2 \beta$ ,  $\sec^2 \gamma$  for its roots.

After making the above substitutions the equation in  $z$  is

$$z^3 - (p^2 + 3)z^2 + (4p^2 - 2pr + 3)z - (p - r)^2 - 1 = 0;$$

$$\therefore \sec^2 \alpha \sec^2 \beta \sec^2 \gamma = \text{product of the roots}$$

$$= (p - r)^2 + 1.$$

Otherwise. Let  $t_1, t_2, t_3$  be the roots of the given equation; then

$$\begin{aligned}
\sec^2 \alpha \sec^2 \beta \sec^2 \gamma &= (1 + t_1^2)(1 + t_2^2)(1 + t_3^2) \\
&= 1 + \Sigma t_1^2 + \Sigma t_1^2 t_2^2 + t_1^2 t_2^2 t_3^2,
\end{aligned}$$

where

$$\Sigma t_1 = p, \quad \Sigma t_1 t_2 = 0, \quad t_1 t_2 t_3 = r.$$

Thus

$$\sec^2 \alpha \sec^2 \beta \sec^2 \gamma = 1 + p^2 - 2pr + r^2.$$

287. Square the given equation, and write it in the form

$$\begin{aligned}
&\sin^2 \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) \sin^2 \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \sin^2 \left( \frac{\pi}{4} - \frac{\gamma}{2} \right) \\
&= \cos^2 \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) \cos^2 \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \cos^2 \left( \frac{\pi}{4} - \frac{\gamma}{2} \right),
\end{aligned}$$

$$\text{or} \quad (1 - \sin \alpha)(1 - \sin \beta)(1 - \sin \gamma) = (1 + \sin \alpha)(1 + \sin \beta)(1 + \sin \gamma);$$

$$\therefore \sin \alpha + \sin \beta + \sin \gamma + \sin \alpha \sin \beta \sin \gamma = 0,$$

$$\text{or} \quad 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} + 8 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} = 0;$$

$$\text{hence} \quad 1 + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 0,$$

$$\text{that is,} \quad 1 + \frac{1}{2}(\cos \alpha + \cos \beta + \cos \gamma - 1) = 0;$$

$$\therefore \cos \alpha + \cos \beta + \cos \gamma + 1 = 0.$$

288. Each side of the heptagon subtends an angle  $\frac{\pi}{7}$  at the circumference of the circle whose diameter is 2. Therefore if  $x$  represent a side,

$$x = 2 \sin \frac{\pi}{7}.$$

Now by Art. 331, the roots of the equation

$$8y^3 + 4y^2 - 4y - 1 = 0,$$

$$\text{are} \quad \cos \frac{2\pi}{7}, \quad \cos \frac{4\pi}{7}, \quad \cos \frac{6\pi}{7}.$$

$$\text{Therefore } 2 \cos \frac{2\pi}{7} \text{ satisfies } y^3 + y^2 - 2y - 1 = 0.$$

$$\text{Put } y = 2 - 4 \sin^2 \frac{\pi}{7} = 2 - x^2 \text{ in this equation.}$$

We obtain, after reduction,

$$x^6 - 7x^4 + 14x^2 - 7 = 0,$$

$$\text{the roots of which are } 2 \sin \frac{\pi}{7}, \quad 2 \sin \frac{2\pi}{7}, \quad 2 \sin \frac{3\pi}{7}.$$

The first of these values corresponds to a side of the heptagon, the second and third to chords subtending at the circumference angles of  $\frac{2\pi}{7}$  and  $\frac{3\pi}{7}$  respectively. That is they represent the diagonals of the heptagon, as is easily seen from a figure.

$$289. \text{ We have } \cot(A + C) = -\cot B = -1;$$

$$\therefore \frac{\cot A \cot C - 1}{\cot A + \cot C} = -1;$$

that is,

$$1 + \cot A + \cot C + \cot A \cot C = 2,$$

or

$$(1 + \cot A)(1 + \cot B) = 2.$$

290. We have  $8R^2 = a^2 + b^2 + c^2$   
 $= (2R \sin A)^2 + (2R \sin B)^2 + (2R \sin C)^2$ ;  
 $\therefore \sin^2 A + \sin^2 B + \sin^2 C = 2$ ;  
 $\therefore 1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C = 4$ ,  
 $\cos 2A + \cos 2B + \cos 2C + 1 = 0$ ;  
 $\therefore 4 \cos A \cos B \cos C = 0$ ; [Examples XII. d. 9.]

therefore one of the angles must be a right angle.

291. We have  $\sin B = \cos A \cos C$   
 $= \cos A \tan C$ ,  
 and  $\cos B = \tan C$ ;  
 $\therefore 1 = \tan^2 C (1 + \cos^2 A)$ ;  
 $\cos^2 C = (1 - \cos^2 C) (1 + \cos^2 A)$ ,  
 $\frac{\sin^2 A}{\cos^2 A} = \left(1 - \frac{\sin^2 A}{\cos^2 A}\right) (2 - \sin^2 A)$ ;  
 or  $\sin^2 A = (1 - 2 \sin^2 A) (2 - \sin^2 A)$ ;  
 whence  $\sin^4 A - 3 \sin^2 A + 1 = 0$ .  
 $\therefore \sin^2 A = \frac{3 - \sqrt{5}}{2}$

the other value being impossible;

$$\therefore \sin^2 A = \frac{1}{4} (6 - 2\sqrt{5}),$$

or  $\sin A = \frac{\sqrt{5} - 1}{2} = 2 \sin 18^\circ$   
 $= \sin B = \sin C$  similarly.

292. It will be sufficient to prove that

$$ab + bc + ca = r^2 + s^2 + 4Rr \dots\dots\dots(1),$$

and  $abc = 4Rrs \dots\dots\dots(2).$

Now  $r^2 + s^2 + 4Rr = r(r_1 + r_2 + r_3) + s^2$  [XVIII. a. Ex. 25]

$$= \left( \frac{\Delta^2}{s(s-a)} + \dots + \dots \right) + s^2$$

$$= \{ (s-b)(s-c) + \dots + \dots \} + s^2$$

$$= 4s^2 - 2s(a+b+c) + bc + ca + ab$$

$$= bc + ca + ab.$$

Again,  $4Rrs = \frac{abc}{\Delta} \cdot \Delta = abc.$

293. We have  $\sin \frac{\pi}{14} = \cos \left( \frac{\pi}{2} - \frac{\pi}{14} \right) = \cos \frac{3\pi}{7}$ ,

which is a root of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

See solution of XXV. c. Ex. 16.

294. The distances of the successive heaps from the starting point are

$$2r \sin \frac{\pi}{n}, \quad 2r \sin \frac{2\pi}{n}, \quad 2r \sin \frac{3\pi}{n}, \quad \dots, \quad 2r \sin \frac{(n-1)\pi}{n};$$

$\therefore$  the whole distance traversed is twice the sum of this series.

Now the sum of the sines =  $\frac{\sin \frac{(n-1)\pi}{2n} \sin \frac{1}{2} \left\{ \frac{\pi}{n} + \frac{(n-1)\pi}{n} \right\}}{\sin \frac{\pi}{2n}}$  [Art. 296.]

$$= \frac{\sin \left( \frac{\pi}{2} - \frac{\pi}{2n} \right) \sin \frac{\pi}{2}}{\sin \frac{\pi}{2n}} = \cot \frac{\pi}{2n},$$

whence the required result follows.

295.  $\cos \frac{\pi}{15} \cos \frac{4\pi}{15} = \frac{1}{2} \left( \cos \frac{\pi}{3} + \cos \frac{\pi}{5} \right) = \frac{1}{2} \left( \frac{1}{2} + \frac{\sqrt{5}+1}{4} \right) = \frac{3+\sqrt{5}}{8};$

$$\cos \frac{2\pi}{15} \cos \frac{7\pi}{15} = \frac{1}{2} \left( \cos \frac{\pi}{3} + \cos \frac{3\pi}{5} \right) = \frac{1}{2} \left( \frac{1}{2} - \frac{\sqrt{5}-1}{4} \right) = \frac{3-\sqrt{5}}{8};$$

$$\cos \frac{3\pi}{15} \cos \frac{6\pi}{15} = \frac{1}{2} \left( \cos \frac{\pi}{5} + \cos \frac{3\pi}{5} \right) = \frac{1}{2} \left( \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} \right) = \frac{1}{4};$$

$$\cos \frac{5\pi}{15} = \cos \frac{\pi}{3} = \frac{1}{2};$$

$\therefore$  multiplying these results together, we have

$$\frac{1}{8} \left( \frac{3+\sqrt{5}}{8} \right) \left( \frac{3-\sqrt{5}}{8} \right) = \frac{4}{8^3} = \frac{2^2}{2^9} = \left( \frac{1}{2} \right)^7.$$

296. We have  $ax + by + cz = 2\Delta.$

Now  $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2$   
 $= (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2;$

or  $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 4\Delta^2 = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2.$

$\therefore x^2 + y^2 + z^2$  is a minimum when the expression on the right is zero; that is when

$$bx = ay, \quad cy = bz, \quad az = cx;$$

$$\therefore \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}.$$

297. With the notation and fig. of Art. 231, suppose  $AD$  and  $BC$  intersect in  $E$ , then  $r_a$  is the radius of the escribed circle opposite to  $E$  in the  $\triangle ABE$ ;

$$\therefore r_a = \frac{a \cos \frac{A}{2} \cos \frac{B}{2}}{\sin \frac{A+B}{2}},$$

$$\therefore \frac{a}{r_a} = \tan \frac{A}{2} + \tan \frac{B}{2};$$

$$\begin{aligned} \therefore \frac{a}{r_a} + \frac{c}{r_c} &= \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} + \tan \frac{D}{2} \\ &= \frac{b}{r_b} + \frac{d}{r_d}, \text{ similarly.} \end{aligned}$$

298. Draw  $AH, AH'$  perpendicular to  $BC$ ; then

$$\angle PAH = (90^\circ - B) - (90^\circ - C) = C - B,$$

and

$$AH = 2R \sin B \sin C,$$

$$\therefore AP = \frac{2R \sin B \sin C}{\cos(C - B)};$$

$$\begin{aligned} \therefore \frac{1}{AP} + \dots + \dots &= \frac{1}{2R} \left\{ \frac{\cos(B - C)}{\sin B \sin C} + \dots + \dots \right\} \\ &= \frac{1}{4R} \cdot \frac{(\sin 2C + \sin 2B) + \dots + \dots}{\sin A \sin B \sin C} \\ &= \frac{2}{R}. \quad [\text{Art. 135, Ex. 1.}] \end{aligned}$$

Again,  $BA' = 2R \cos BA'A = 2R \cos C$ , since  $B, A', A, C$  are concyclic,

$$\therefore A'H' = 2R \cos B \cos C;$$

$$\begin{aligned} \therefore \frac{1}{A'P} + \dots + \dots &= \frac{1}{2R} \left\{ \frac{\cos(C - A)}{\cos B \cos C} + \dots + \dots \right\} \\ &= \frac{1}{2R} \cdot \frac{(\cos 2C + \cos 2B) + \dots + \dots}{\cos A \cos B \cos C} \\ &= \frac{2(\cos 2A + \cos 2B + \cos 2C)}{4R \cos A \cos B \cos C} \\ &= \frac{1}{2R} \cdot \frac{4 \cos A \cos B \cos C + 1}{\cos A \cos B \cos C}. \quad [\text{XII. d. Ex. 9.}] \end{aligned}$$

299. Let  $\tan \frac{\theta}{2} = t$ , then  $\sin \theta = \frac{2t}{1+t^2}$ ,  $\cos \theta = \frac{1-t^2}{1+t^2}$ ; substitute these values in the given equation; then after reduction, we obtain

$$t^4b + 2t^3(c-a) - 2t(c+a) - b = 0.$$

This equation has *four* roots; three of which are

$$\tan \frac{\alpha}{2}, \tan \frac{\beta}{2}, \tan \frac{\gamma}{2};$$

also  $t_1 t_2 t_3 t_4 = -1$ ,  $t_2 t_3 + t_3 t_4 + \dots = 0$ .

Eliminating  $t_4$  by means of these equations,

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = \frac{1}{t_2 t_3} + \frac{1}{t_3 t_1} + \frac{1}{t_1 t_2};$$

$$\therefore \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} + \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$$

$$= \cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2};$$

that is,

$$\Sigma \left( \tan \frac{\beta}{2} \tan \frac{\gamma}{2} - \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \right) = 0;$$

$$\therefore \Sigma \frac{\sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} - \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}}{\sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}} = 0;$$

$$\therefore \Sigma \cos \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 0;$$

$$\therefore \Sigma (\cos \beta + \cos \gamma) \sin \alpha = 0;$$

or

$$\Sigma \sin (\alpha + \beta) = 0.$$

300. By Example 2, Art. 331,

$$\sec^2 \frac{\pi}{7}, \quad \sec^2 \frac{2\pi}{7}, \quad \sec^2 \frac{3\pi}{7}$$

are roots of the equation

$$x^3 - 24x^2 + 80x - 64 = 0;$$

$$\therefore \sec^2 \frac{\pi}{7} + \sec^2 \frac{2\pi}{7} + \sec^2 \frac{3\pi}{7} = 24.$$

Also from XXV. c. Ex. 21,

$$\operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7} = 8,$$

whence the required result follows.

